Trigonometric Identities

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1 Prerequisites

You should know the cosine and sine of 0, $\pi/6$, $\pi/4$, $\pi/3$, and $\pi/2$. Memorize these if you do not already know them.

$$\begin{aligned}
\cos 0 &= 1 & \sin 0 &= 0 \\
\cos\frac{\pi}{6} &= \frac{\sqrt{3}}{2} & \sin\frac{\pi}{6} &= \frac{1}{2} \\
\cos\frac{\pi}{4} &= \frac{1}{\sqrt{2}} & \sin\frac{\pi}{4} &= \frac{1}{\sqrt{2}} \\
\cos\frac{\pi}{3} &= \frac{1}{2} & \sin\frac{\pi}{3} &= \frac{\sqrt{3}}{2} \\
\cos\frac{\pi}{2} &= 0 & \sin\frac{\pi}{2} &= 1
\end{aligned}$$

You should also be able to graph cosine and sine. These graphs should exist in your head, so that you can use them to reason, and to "see" why some things work the way they do. It is not enough that your calculator or computer can graph these functions. You must be able to bring up these graphs instantly in your head when you need them.

You should know that cosine is an even function, meaning

$$\cos(-\theta) = \cos\theta,$$

and that sine is an odd function, meaning

$$\sin(-\theta) = -\sin\theta.$$

The evenness of cosine and the oddness of sine can be seen from their graphs.

2 The Euler Formula

Every trigonometric identity that I am aware of can be derived from the Euler formula,

$$e^{i\theta} = \cos\theta + i\sin\theta, \tag{1}$$

which holds for all real numbers (or real angles in radians) θ.

Problem 1 Given the Euler formula (1), show that

$$e^{-i\theta} = \cos\theta - i\sin\theta. \tag{2}$$

3 Relationships between Cosine and Sine

Using $\theta = \pi/2$ in the Euler formula (1), we see that

$$e^{i\frac{\pi}{2}} = i. \tag{3}$$

Multiplying this equation by the Euler formula (1) gives

$$e^{i\frac{\pi}{2}}e^{i\theta} = i\cos\theta - \sin\theta.$$

Taking the real part of each side gives

$$\cos(\theta + \frac{\pi}{2}) = -\sin\theta. \tag{4}$$

You can see this identity in your head if you know the graphs of cosine and sine. The left side of (4) is the cosine function shifted to the *left* by $\pi/2$. (We know it's shifted to the left because the peak occurs at $\theta = -\pi/2$.) The right side of the identity is the sine function flipped upside down. Take a few minutes and see if you can see this identity in your head.

Problem 2 Use similar techniques to show that

$$\sin(\theta + \frac{\pi}{2}) = \cos\theta \tag{5}$$

$$\cos(\theta - \frac{\pi}{2}) = \sin\theta \tag{6}$$

$$\sin(\theta - \frac{\pi}{2}) = -\cos\theta \tag{7}$$

See if you can see these identities in your head by manipulating the graphs of cosine and sine.

Next, multiply (3) by (2), and take the real part to get

$$\cos(\frac{\pi}{2} - \theta) = \sin\theta. \tag{8}$$

The left side is the same as $\cos(\theta - \pi/2)$, since cosine is even. Try to visualize this identity with graphs.

Problem 3 Use a similar technique to show that

$$\sin(\frac{\pi}{2} - \theta) = \cos\theta \tag{9}$$

Try to visualize this identity with graphs.

4 Sum and Difference Formulas

The sum and difference formulas can be derived from the observation that

$$e^{i(A+B)} = e^{iA}e^{iB}. \tag{10}$$

Expanding the left side using the Euler formula gives

$$e^{i(A+B)} = \cos(A+B) + i\sin(A+B),$$

while the right side gives

$$\begin{aligned} e^{iA}e^{iB} &= (\cos A + i\sin A)(\cos B + i\sin B) \\ &= \cos A\cos B - \sin A\sin B + i(\sin A\cos B + \cos A\sin B) \end{aligned}$$

Equating the real parts of the left and right sides of (10) gives

$$\cos(A+B) = \cos A\cos B - \sin A\sin B, \tag{11}$$

which is the identity for the cosine of the sum of two angles. Equating the imaginary parts of the left and right sides of (10) gives

$$\sin(A+B) = \sin A\cos B + \cos A\sin B \tag{12}$$

which is the identity for the sine of the sum of two angles.

If we now want identities for the difference of two angles, we can simply replace B with $-B$ in equations (11) and (12) above, and use the fact the cosine is an even function,

$$\cos(-B) = \cos B,$$

while sine is an odd function,

$$\sin(-B) = -\sin B.$$

This gives identities for the difference between two angles.

$$\cos(A - B) = \cos A \cos B + \sin A \sin B \tag{13}$$

$$\sin(A - B) = \sin A \cos B - \cos A \sin B \tag{14}$$

5 Unit Circle Identity

If we take $A = \theta$ and $B = \theta$ in (13), we get the "unit circle identity"

$$\cos^2 \theta + \sin^2 \theta = 1. \tag{15}$$

6 Double Angle Formulas

If we take $A = \theta$ and $B = \theta$ in (11), we get

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta, \tag{16}$$

which is the double angle formula for cosine. Using the unit circle identity, we can also write

$$\cos 2\theta = 2\cos^2 \theta - 1 \tag{17}$$

$$\cos 2\theta = 1 - 2\sin^2 \theta. \tag{18}$$

If we take $A = \theta$ and $B = \theta$ in (12), we get

$$\sin 2\theta = 2 \sin \theta \cos \theta, \tag{19}$$

which is the double angle formula for sine.

7 Help Eliminate Errors

Accuracy and correctness are very important to me, as well as to readers of this document. If this document contains errors, I would like to fix them, and you can help me. Please let me know if you find spelling mistakes, mathematical mistakes, or any other mistakes. In honor of Donald Knuth, who set the standard for care in technical writing, I will happily pay \$2.56 (one hexadecimal dollar) for each substantive or typographical error in this document to the first person who reports the error.