Electromagnetic Waves

Scott N. Walck

January 13, 2025

Contents

Recall the Maxwell equations.

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \qquad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$$

$$\nabla \times \mathbf{B} - \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{J} \qquad \nabla \cdot \mathbf{B} = 0$$

Problem 1. Give the names for each of the Maxwell equations.

Problem 2. For each of the following items, give its name, give its units, and classify the item as a scalar field, a vector field, or something else.
(a) $\nabla \cdot$
(b) $\mathbf{E}$
(c) ρ
(d) ϵ_0
(e) $\nabla \times$
(f) $\mathbf{B}$
(g) μ_0
(h) t
(i) $\mathbf{J}$

Our attitude at the moment is to imagine that we know the charge and current distributions, and we want to solve for the electric and magnetic fields. This attitude leads us to think of the Maxwell equations as a set of coupled partial differential equations.

Problem 3. Why are they partial differential equations, and not ordinary differential equations? Why are they coupled?

Problem 4. Write out the Maxwell equations in Cartesian coordinates as a set of coupled partial differential equations. Use the components E_x, E_y, and E_z rather than the vector $\mathbf{E}$. Use the coordinates x, y, and z, and get rid of the ∇ symbol. How many equations are there in all? What are the independent variables? What are the dependent variables? Are the Maxwell equations linear? Homogeneous? First order? Second order?

Problem 5. Are the Maxwell equations "better" than Coulomb's law for describing electricity? If so, how?

Problem 6. Can you name some ways in which electricity and magnetism are related?

In these notes, we are interested in the properties of electromagnetic (EM) waves. The Maxwell equations can be used to show how accelerating charges and oscillating currents create EM waves. The Maxwell equations can also be used to show how EM waves interact with matter, for example by reflecting, refracting, and scattering. Our goals for these notes are more fundamental. The most basic thing that a wave can do (more basic than being created, or scattering, or getting absorbed) is to propagate. These notes are principally about the *propagation* of EM waves. Notably, the Maxwell equations also describe the propagation of EM waves. The simplest propagation is through empty space, where there is no matter, no charges, and no currents.

For the first part of these notes, we are interested in solutions to the Maxwell equations in free space, that is, in regions where there are no charges and no currents.

Problem 7. Write down the free-space Maxwell equations.

Problem 8. Find the simplest solution to the free-space Maxwell equations.

Problem 9. Find another solution to the free-space Maxwell equations.

In order for a solution of the free-space Maxwell equations to be regarded as a wave, it needs to oscillate in space and it needs to oscillate in time.

Electromagnetic waves are rather complicated waves, because we have two vector quantities (the electric and magnetic fields) that depend on three space dimensions and a time dimension. Before we search for wave-like solutions to the free-space Maxwell equations, let us digress to think about waves in a simpler setting.

1 Waves in One Spatial Dimension

1.1 Scalar waves in one space dimension

Let's take a pressure wave as an example of a scalar wave. A pressure wave is a scalar wave because pressure is a scalar quantity. To start, let's make things even simpler and work in a fantasy world that has only one space dimension x and one time dimension t. Our pressure wave is described by a function $P(x,t)$. This function tells you the pressure at position x and time t. The function P will satisfy a wave equation

$$\frac{\partial^2 P}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 P}{\partial t^2} = 0, \tag{1}$$

where v is a constant with units of velocity that depends on the properties of the fluid through which the pressure wave propagates. Every function that describes a wave satisfies a partial differential equation like this. (Ordinary differential equations can be used to describe oscillations, but we need partial differential equations to describe waves.)

Problem 10. Find the simplest solution to the wave equation (1).

Problem 11. Find another solution to the wave equation (1).

In order for a solution of the wave equation (1) to be regarded as a nontrivial wave, it needs to oscillate in space and it needs to oscillate in time.

Our prototypical solution to wave equation (1) is

$$P(x,t) = P_0 \cos(kx - \omega t), \tag{2}$$

where P_0 is a constant with units of pressure, k is called the *wavenumber*, and has units of inverse length, and ω is the angular frequency, with units of radians per unit time.

Notice that $P(x,t)$ can be negative. This is because $P(x,t)$ is a description of the *relative* pressure (also called *gauge* pressure) at position x at time t, and not the absolute pressure. Relative pressure means pressure with respect to the equilibrium pressure of the fluid through which the pressure wave propagates.

Problem 12. What is the wavelength of (2)? Explain how you know.

Problem 13. What is the period of (2)? Explain how you know.

Problem 14. What is the frequency (denoted f or ν) of (2)? Explain how you know.

1.1.1 Points of constant phase

A useful way to think about what is happening with a traveling wave in one spacial dimension, like (2), is to examine the *points of constant phase*. These are the places (values of x) at which the pressure is the same. These places will change in time. For example, suppose we decide to follow wave crests. These are the places where the pressure is maximum, $P(x,t) = P_0$. The cosine in (2) must give one at these places. Therefore, the argument of the cosine function must be a multiple of 2π.

Problem 15. For the wave described in (2), write a relationship between k, x, ω, and t that must hold at wave crests. Use this expression to give the position of a wave crest as a function of time.

Problem 16. What is the velocity of a point on the crest of the wave (2)?

We don't need to look at wave crests only. We could choose to follow wave troughs (in which case we would take $P(x,t) = -P_0$), points of equilibrium pressure (in which case we would take $P(x,t) = 0$), or any other points (say $P(x,t) = P_0/2$, for example). In all of these cases, the points of constant phase have the same velocity.

Problem 17. Show that points on the wave trough of (2) have the same velocity as points on the wave crest.

Because, for a traveling wave like (2), any point of constant phase moves with the same velocity, we will call this the *wave velocity.*

Problem 18. Insert (2) into wave equation (1), and find a relationship between v, k, and ω that needs to hold in order for (2) to be a solution of (1). In light of this relationship, what is the meaning of v in (1)?

Problem 19. Rewrite the relationship you found in the previous problem in terms of wave speed v, wavelength λ, and period T. Give a physical interpretation for this relationship. (In other words, what does it mean?) Also, give the relationship between v, λ, and ν.

Problem 20. In what direction is wave (2) propagating? Explain how you know.

Problem 21. Write an expression for a wave propagating in the opposite direction.

Problem 22. For wave (2), sketch a graph of P as a function of x at a fixed time t. Sketch a second graph of P vs. x at a slightly later time, and a third graph at a still later time.

1.1.2 Phase angle

Wave (2) has maximum pressure at $x = 0$ at $t = 0$. There are solutions to (1) very much like (2) that do not have this attribute.

Problem 23. Show that

$$P(x,t) = P_0 \cos(kx - \omega t + \delta) \tag{3}$$

is a solution to (1) for any real number δ. We call δ a *phase angle*.

Problem 24. Find a phase angle so that (3) is a wave with minimum pressure at $x = 0$, $t = 0$.

Problem 25. Find a phase angle so that (3) is a wave with zero pressure at $x = 0$, $t = 0$. In fact, there are two different waves that have this feature. See if you can find both.

1.1.3 Standing waves

A standing wave is a wave with nodes. A node is a place where the (relative) pressure is zero and remains zero over time (remember zero relative pressure means the equilibrium pressure of the fluid).

Problem 26. At $t = 0$, a pressure wave has nodes at $-3L$, $-2L$, $-L$, 0, L, $2L$, $3L$, etc. Write a function of position x describing the pressure.

Problem 27. Take the function of position from the previous problem and make it into a wave by multiplying it by $\cos \omega t$. Show that the wave satisfies the wave equation (1). Are there conditions that must hold in order for this standing wave to be a solution?

Problem 28. For the standing wave you formed in the previous problem, sketch a graph of P as a function of x at a fixed time t. Sketch a second graph of P vs. x at a slightly later time, and a third graph at a still later time.

Problem 29. What is the wavelength of the standing wave? What is the period of the standing wave?

Problem 30. Do the wavelength, period, and speed v have the same relationship for the standing wave that they do for the traveling wave?

Problem 31. Consider the standing wave

$$P(x,t) = P_0 \sin kx \cos \omega t.$$

Where are the nodes of this standing wave? What condition must hold for this wave to satisfy the wave equation (1)? Write this standing wave as a sum of a traveling wave propagating in the positive x direction and a traveling wave propagating in the negative x direction. (You will need to use some trigonometric identities to do this.)

Problem 32. Consider the standing wave

$$P(x,t) = P_0 \sin kx \sin \omega t.$$

Where are the nodes of this standing wave? What condition must hold for this wave to satisfy the wave equation (1)? Write this standing wave as a sum of a traveling wave propagating in the positive x direction and a traveling wave propagating in the negative x direction.

Problem 33. Consider the standing wave

$$P(x,t) = P_0 \cos kx \cos \omega t.$$

Where are the nodes of this standing wave? What condition must hold for this wave to satisfy the wave equation (1)? Write this standing wave as a sum of a traveling wave propagating in the positive x direction and a traveling wave propagating in the negative x direction.

Problem 34. Consider the standing wave

$$P(x,t) = P_0 \cos kx \sin \omega t.$$

Where are the nodes of this standing wave? What condition must hold for this wave to satisfy the wave equation (1)? Write this standing wave as a sum of a traveling wave propagating in the positive x direction and a traveling wave propagating in the negative x direction.

Now is a good time to do the Function Animation Homework

1.1.4 Numerical solution of the wave equation

Let's return to the one-dimensional scalar wave equation (1), which we repeat here for convenience.

$$\frac{\partial^2 P}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 P}{\partial t^2} = 0$$

To analyze this equation numerically, we discretize both space and time. In space we choose a spacial step Δx that is small compared with important length scales of the situation.

$$\frac{\partial P}{\partial x}(x,t) \approx \frac{P(x+\Delta x/2,t) - P(x-\Delta x/2,t)}{\Delta x}$$

$$\begin{aligned}\frac{\partial^2 P}{\partial x^2}(x,t) &\approx \frac{\frac{\partial P}{\partial x}(x+\Delta x/2,t) - \frac{\partial P}{\partial x}(x-\Delta x/2,t)}{\Delta x} \\ &\approx \frac{P(x+\Delta x,t) - P(x,t) - [P(x,t) - P(x-\Delta x,t)]}{\Delta x^2} \\ &= \frac{P(x+\Delta x,t) - 2P(x,t) + P(x-\Delta x,t)}{\Delta x^2}\end{aligned}$$

In time we choose a time step Δt that is small compared with important time scales of the situation.

$$\frac{\partial^2 P}{\partial t^2}(x,t) \approx \frac{P(x,t+\Delta t) - 2P(x,t) + P(x,t-\Delta t)}{\Delta t^2}$$

Now insert our discrete approximations into the wave equation.

$$\frac{P(x+\Delta x,t) - 2P(x,t) + P(x-\Delta x,t)}{\Delta x^2} = \frac{1}{v^2}\frac{P(x,t+\Delta t) - 2P(x,t) + P(x,t-\Delta t)}{\Delta t^2}$$

Multiply both sides by $v^2\Delta t^2$, then switch sides.

$$P(x,t+\Delta t) - 2P(x,t) + P(x,t-\Delta t) = \frac{v^2\Delta t^2}{\Delta x^2}\left[P(x+\Delta x,t) - 2P(x,t) + P(x-\Delta x,t)\right]$$

Now solve for P at the future-most time in terms of P at the present and in the past.

$$\begin{aligned}P(x,t+\Delta t) &= 2P(x,t) - P(x,t-\Delta t) \\ &\quad + \frac{v^2\Delta t^2}{\Delta x^2}\left[P(x+\Delta x,t) - 2P(x,t) + P(x-\Delta x,t)\right] \\ &= 2\left(1 - \frac{v^2\Delta t^2}{\Delta x^2}\right)P(x,t) - P(x,t-\Delta t) \\ &\quad + \frac{v^2\Delta t^2}{\Delta x^2}\left[P(x+\Delta x,t) + P(x-\Delta x,t)\right]\end{aligned}$$

We need
$$\frac{v\Delta t}{\Delta x} \leq 1$$
for stability. If we violate this inequality, we run the risk of seeing a numerical instability that produces complete trash. The numerical stability of numerical solutions to partial differential equations is an interesting area of numerical analysis that we don't really have time to go into. People who studied this in detail found that we get the smallest error if we hug the stability threshold and take
$$\frac{v\Delta t}{\Delta x} = 1.$$
(I should really provide a reference for this assertion. Sorry.) The the previous equation simplifies to

$$P(x, t + \Delta t) = P(x + \Delta x, t) + P(x - \Delta x, t) - P(x, t - \Delta t). \tag{4}$$

1.1.5 Boundary Conditions

When we work analytically with waves in one spatial dimension, we consider the entire real line of all x values as our space. The variable x ranges over the entire region $-\infty < x < \infty$.

When we do numerical work with a finite spatial step Δx and a finite time step Δt, we are typically paying attention to only a finite length of one-dimensional space. The values of x are limited by boundaries at the left and the right. The variable x may only range over an interval like $0 < x < L$ for some length L.

This raises the question of what to do at the left and right boundaries. Suppose that our left boundary occurs at $x = 0$. The update equation

$$P(x, t + \Delta t) = P(x + \Delta x, t) + P(x - \Delta x, t) - P(x, t - \Delta t)$$

tells us that we need the value of P at $x - \Delta x = -\Delta x$, which is to the left of zero, and hence outside the region we are keeping track of. We don't have a value to plug into the equation. So, we need to make a decision about what to do. This is the problem of "boundary conditions" as they apply to the numerical solution of partial differential equations.

One simple choice is to fix the boundaries at $P = 0$, in other words to demand that the value of the dependent variable (P in this case) must take the value zero at the boundaries. This is a convenient choice, and has the effect of making waves reflect when they get to the boundary. It it also a useful choice for standing waves. It is not a great choice for a traveling wave.

A second choice is to decide what values the boundary will have, and let those values change in time. In this way, we can simulate a source of the wave, like shaking the end of a string up and down.

A third choice is to use period boundary conditions. In this case, when a value is needed that is too far to the right, we pretend that the one-dimensional space wraps around and we use a value from the left. Similarly, when we need a value too far to the left, we take it from the right. This is a good choice for traveling waves.

A fourth choice is to let the boundaries grow as the calculation evolves. We assume that $P = 0$ anywhere where we haven't calculated it yet, but we let the spatial grid grow by one step on the left and one step on the right at every time step. A wave that is produced by a source will continue to travel without reflecting or wrapping around to the other side of the space.

If you get deeper into this subject of the numerical analysis of partial differential equations, you will find that there are many choices, some subtle and complex, for what to do at the boundaries. We do not have time to go into these details. We will use one of the options mentioned above.

1.1.6 The inhomogeneous wave equation

An *inhomogenous wave equation* is a wave equation with a source term. It has the form

$$\frac{\partial^2 P}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 P}{\partial t^2} = f, \tag{5}$$

where f is a *source term* that can depend on x and t. The function f is a called a source term because it can create waves where there were no waves before. In the Maxwell equations, charge and current density serve as source terms for the electric and magnetic fields. Charge and current density can create electric and magnetic waves where there were no waves before, and they can modify the behavior of waves that currently exist. Mechanical vibrations can serve as a source term for a pressure wave. Mechanical vibrations can create a pressure wave where there was none before.

Problem 35. Repeat the derivation of equation (4) in section 1.1.4, starting with equation (5) instead of equation (1). You should end up with an equation that looks a lot like equation (4), but also includes the source term in some way.

An inhomogeneous wave equation is capable of expressing both the propagation of a wave and the creation of a wave. The homogeneous wave equation that we studied first only expresses the propagation of a wave.

1.1.7 Computational details

In preparing to solve the 1D wave equation with a computer, there are several quantities we need to think about that do not come into play when solving the wave equation analytically. Two that we have already introduced are the time step Δt and the spatial step size Δx. The time step Δt needs to be small compared with important time scales in the problem of interest. If the wave of interest has a well-defined period T, then the time step must be much smaller than the wave period. We must choose

$$\Delta t \ll T$$

or we cannot expect the results of the computer to be very accurate. If we do not know the period T, we can take a guess at the time step Δt and see what we get. Sometimes it's obvious that the results are wrong and that we need a smaller time step.

In principle, there is no harm in choosing a time step that is too small, but in practice there is harm. For example, suppose we knew that the period of a wave was about one second. We could choose a time step of one microsecond, but then we would need the computer to go through one million sets of calculating the dependent variable (pressure in our examples above) at every point in space, and this might take the computer longer than we want to wait. As a rough rule, try to pick a time step that is about 1% of the period, or about 1% of any other time scale in which quantities change in important ways.

We said above that to minimize numerical errors, it is good to choose the spatial step size to be $\Delta x = v\Delta t$, where v is the wave velocity. We often know the wave velocity; for EM waves it is the speed of light. For other waves, it is different, but we usually know it. So, we are not choosing Δt and Δx independently. They are linked by the wave velocity.

Two additional computational parameters we need to choose are the number of spatial steps we will use to keep track of our pressure, and the number of time steps over which we will run our simulation. In the code that I am sharing with you, I think I chose 200 for the number of spatial locations, and 300 for the number of time steps. There is nothing magic about these numbers. We could make them larger if we have the patience to wait for the computer to calculate the results. For most of the 1D calculation techniques we study, the time needed by the computer is linear in each of these step numbers. If we double the number of time steps, the simulation will take about twice as long. If we double the number of spatial steps, the simulation will also take about twice as long.

How small can we make the number of spatial steps? If there is a wavelength for our wave, the

So far we have four computational parameters: Δt, Δx, the number of time steps, and the number of spatial steps. Only three of these parameters can be chosen independently, because we want to enforce $\Delta x = v\Delta t$.

We need to choose all sorts of values when we ask the computer to deal with a wave. Let's make a list of the values the computer needs to know. There is Δx, Δt, v, λ, ω, k, the number of spatial steps, the number of time steps, and maybe other pieces of information.

Now is a good time to do the 1D Scalar Wave Animation Homework

(traveling wave with no source, standing wave with no source, shake the left end, source in the middle with space growing) Each of the 4 boundary conditions gets used once.

1.2 EM waves in one space dimension

Problem 36. In Problem 4, we wrote the Maxwell equations in Cartesian coordinates. There were eight equations in all. Start with these equations and suppose that nothing depends on y or z. This implies that any partial derivatives with respect to y or z must vanish. Write a set of eight simpler equations for this situation.

Problem 37. There are six dependent variables in the eight equations you just wrote down: E_x, E_y, E_z, B_x, B_y, and B_z. In the Maxwell equations of Problem 4, all six were coupled together.

In this newest set of eight equations, the dependent variables partially uncouple from each other. Place each dependent variable into a bag so that it only appears in equations with other dependent variables in that bag. How many bags are there, and who is in each bag?

Problem 38. Each of the eight one-spatial-dimension Maxwell equations can be placed into one of the bags you defined in the previous problem. Each equation goes in the bag in which the dependent variables in the equation live. Show how the eight Maxwell equations get divided up.

Problem 39. In problem 38, you should have found one bag that contains two equations in which the only dependent variable is B_x. Use these two equations to find the general solution for $B_x(x,t)$. You may give this solution in terms of one or more constants that you make up names for.

Problem 40. In problem 38, you should have found two bags, each of which contains two different dependent variables. (So, four different dependent variables in all.) The equations in one bag are very similar to the equations in the other bag. Starting with the equations in one bag (either one), try to replace the dependent variables in the first bag with the dependent variables in the second bag in just the right way so that you obtain the equations in the second bag. If you succeed in this, then you have shown that the two sets of equations are mathematically identical, up to the names of the dependent variables. If we figure out how to solve one set of equations, then we automatically have a way to solve the second set.

Suppose we want to work with the equations that contain E_y and B_z. Some people call this set of two equations the 1D TE mode. It is decoupled from the 1D TM mode, which contains the two equations with E_z and B_y. Because these two sets of equations are decoupled from each other, we can solve each set independently.

Problem 41. Discretize space and time, as we did before for the homogeneous and inhomogeneous wave equations, now for the two equations that make up the 1D TE mode. Find expressions for E_y and B_z at a later time in terms of their values at earlier times, along with the value of any sources at any time.

Notice from the result of the previous problem that we are not keeping track of E_y and B_z at the same locations in space and time. These two components are staggered with respect to each other. The following table tries to give a picture of what this looks like.

	$x-\Delta x$	$x-\Delta x/2$	x	$x+\Delta x/2$	$x+\Delta x$
$t-\Delta t$		B_z		B_z	
$t-\Delta t/2$	E_y		E_y		E_y
t		B_z		B_z	
$t+\Delta t/2$	E_y		E_y		E_y
$t+\Delta t$		B_z		B_z	

The staggered location of dependent variables is not something we needed to deal with for the scalar wave equation. It is a consequence of our using a "balanced derivative" (forward and backward by a half step) and having more than one dependent variable.

Now is a good time to do the 1D EM Wave Animation Homework

2 Waves in Two Spatial Dimensions

2.1 Scalar waves in two space dimensions

Let's continue to consider a pressure wave as an example of a scalar wave. Let's make things slightly richer by working in a world that has two space dimensions x and y, and one time dimension t. Our pressure wave is now described by a function $P(x, y, t)$. This function tells you what the pressure is at position (x, y) and time t. The function P will satisfy a wave equation

$$\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} - \frac{1}{v^2}\frac{\partial^2 P}{\partial t^2} = 0, \tag{6}$$

where v is a constant with units of velocity that depends on the properties of the fluid through which the pressure wave propagates.

Our favorite solution to (6) is

$$P(x, y, t) = P_0 \cos(k_x x + k_y y - \omega t + \delta). \tag{7}$$

Problem 42. Insert (7) into (6) and find any conditions that must hold in order for (7) to be a solution to (6).

We wish to understand the meaning of k_x and k_y in (7). To help us understand that, we will look at the lines of constant phase.

2.1.1 Lines of constant phase

In two spacial dimensions, a wave like (7) has lines of constant phase rather than points of constant phase.

Problem 43. For the wave described in (7), write the equation of a line in the xy plane that describes one of the wave crests at time t.

Problem 44. Now write the equation of a line in the xy plane that describes one of the wave crests at time $t + \Delta t$. Sketch the line of wave crests at time t in the xy plane and the line of wave crests at time $t + \Delta t$ on the same picture.

Problem 45. Find the distance between these two lines in the xy plane.

Problem 46. Use the result of the previous problem to find the speed of the wave.

Problem 47. Find the lines of constant phase for the wave (7) with $k_x = \frac{1}{\sqrt{2}}\frac{\omega}{v}$, $k_y = \frac{1}{\sqrt{2}}\frac{\omega}{v}$, and $\delta = 0$.

Problem 48. What is the direction of propagation for the wave in the previous problem?

Problem 49. Find the lines of constant phase for the wave (7) with $k_x = \frac{\omega}{v}$, $k_y = 0$, and $\delta = 0$. Give the direction of propagation.

Problem 50. Find the lines of constant phase for the wave (7) with $k_x = 0$, $k_y = \frac{\omega}{v}$, and $\delta = 0$. Give the direction of propagation.

Problem 51. What can you say about the relationship between the lines of constant phase and the direction of propagation?

Problem 52. What can you say about the relationship between the vector $k_x\hat{\mathbf{i}} + k_y\hat{\mathbf{j}}$ (called the wave vector) and the direction of propagation?

Problem 53. Write a pressure wave in 2D traveling in the negative x direction, with angular frequency ω, amplitude P_0, and wave speed v.

Problem 54. Write a pressure wave in 2D traveling in the negative y direction, with wavelength λ, amplitude P_0, and wave speed v.

A wave is any solution to a wave equation. A traveling wave is a wave with a well-defined direction of propagation.

Problem 55. Give an example of a 2D wave without a well-defined direction of propagation.

Problem 56. Give an example of a 1D wave without a well-defined direction of propagation.

Problem 57. Sketch a 2D traveling wave (P as a function of x and y) at t, $t + \Delta t$, and $t + 2\Delta t$.

2.1.2 Standing waves in 2D

Problem 58. Is

$$P(x, y, t) = P_0 \sin k_x x \sin k_y y \cos \omega t$$

a solution to wave equation (6)? If so, describe its nodes.

Problem 59. Is

$$P(x, y, t) = P_0 \sin k_x x \cos(k_y y - \omega t)$$

a solution to wave equation (6)? If so, describe its nodes.

Problem 60. Write an expression for a pressure wave $P(x, y, t)$ with amplitude P_0, wave velocity v and the following nodes. The lines $x = 0$ m, $x = 2$ m, $x = -2$ m, $x = 4$ m, $x = -4$ m, and so on are nodes. The lines $y = 0$ m, $y = 3$ m, $y = -3$ m, $y = 6$ m, $y = -6$ m, and so on are nodes.

2.1.3 Linearity

Wave equations (1) and (6) are *linear*, *homogeneous* partial differential equations. This means that the sum of any two (or more) solutions is also a solution. In fact, we can build up all possible solutions as sums of traveling wave solutions.

2.2 EM waves in two space dimensions

Problem 61. In Problem 4, we wrote the Maxwell equations in Cartesian coordinates. There were eight equations in all. Start with these equations and suppose that nothing depends on z. This implies that any partial derivatives with respect to z must vanish. Write a set of eight simpler equations for this situation.

Problem 62. There are six dependent variables in the eight equations you just wrote down: E_x, E_y, E_z, B_x, B_y, and B_z. In the Maxwell equations of Problem 4, all six were coupled together. On the other hand, in the one-spatial-dimension Maxwell equations, where nothing depends on y or z, the six dependent variables partially decoupled into four bags: the first bag contained E_x, the second bag contained B_x, the third bag contained E_y and B_z, and the fourth bag contained E_z and B_y.

In this newest set of eight equations, the dependent variables again partially uncouple from each other, but not as much as in the case of one spatial dimension. Place each dependent variable into a bag so that it only appears in equations with other dependent variables in that bag. How many bags are there, and who is in each bag?

Problem 63. Each of the eight two-spatial-dimension Maxwell equations can be placed into one of the bags you defined in the previous problem. Each equation goes in the bag in which the dependent variables in the equation live. Show how the eight Maxwell equations get divided up.

2.2.1 Numerical Solution of the 2D TM Mode

$$\frac{\partial B_x}{\partial t} = -\frac{\partial E_z}{\partial y} \tag{8}$$

$$\frac{\partial B_y}{\partial t} = \frac{\partial E_z}{\partial x} \tag{9}$$

$$\frac{\partial E_z}{\partial t} = c^2\frac{\partial B_y}{\partial x} - c^2\frac{\partial B_x}{\partial y} - \frac{1}{\epsilon_0}J_z \tag{10}$$

$$\frac{B_x(x,y,t+\Delta t/2) - B_x(x,y,t-\Delta t/2)}{\Delta t} = -\frac{E_z(x,y+\Delta y/2,t) - E_z(x,y-\Delta y/2,t)}{\Delta y}$$

$$\frac{B_y(x,y,t+\Delta t/2) - B_y(x,y,t-\Delta t/2)}{\Delta t} = \frac{E_z(x+\Delta x/2,y,t) - E_z(x-\Delta x/2,y,t)}{\Delta x}$$

$$\frac{\partial E_z}{\partial t} = c^2\frac{\partial B_y}{\partial x} - c^2\frac{\partial B_x}{\partial y} - \frac{1}{\epsilon_0}J_z$$

3 Waves in Three Spatial Dimensions

3.1 Scalar waves in three space dimensions

Let's continue to consider a pressure wave as an example of a scalar wave. We now work in a world that has three space dimensions x, y, and z, and one time dimension t. Our pressure wave is now described by a function $P(x, y, z, t)$. This function tells you what the pressure is at position (x, y, z) and time t.

Problem 64. Write down the wave equation that $P(x, y, z, t)$ will satisfy. Use Cartesian coordinates.

Problem 65. Rewrite the wave equation that $P(\mathbf{r}, t)$ will satisfy, using the Laplacian operator ∇^2. Put it in the box below. This wave equation is now expressed in a coordinate-free manner, because it does not make explicit reference to the Cartesian coordinates x, y, and z.

(11)

Problem 66. In analogy with (3) and (7), write down the general traveling wave solution to (11) in Cartesian coordinates.

Problem 67. Rewrite the solution in a coordinate-free manner, in terms of the vector $\mathbf{k}$, the vector $\mathbf{r}$, and the dot product. Put it in the box below.

(12)

Problem 68. Give the conditions that must hold for (12) to be a solution to (11).

3.1.1 Planes of constant phase

In three spacial dimensions, a wave like (12) has planes of constant phase rather than points or lines of constant phase.

Problem 69. Find the planes of constant phase (lets choose wave crests, to be concrete) for the wave (12) with

$$\mathbf{k} = \frac{\omega}{v}\left(\frac{1}{\sqrt{2}}\hat{\mathbf{j}} + \frac{1}{\sqrt{2}}\hat{\mathbf{k}}\right)$$

and $\delta = 0$.

Problem 70. What is the direction of propagation for the wave in the previous problem?

Problem 71. What can you say about the relationship between the planes of constant phase and the direction of propagation?

Problem 72. What can you say about the relationship between the vector $\mathbf{k}$ and the direction of propagation?

Problem 73. Write a pressure wave in 3D traveling in the direction $\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}}$ with angular frequency ω, amplitude P_0, and wave speed v.

Problem 74. Write a pressure wave in 3D traveling in the negative y direction, with period T, amplitude P_0, and wave speed v.

4 EM Waves

4.1 EM Wave Equation

The Maxwell equations do not look like a wave equation. A wave equation is a partial differential equation that has second derivatives in space and a second derivative in time. The Maxwell equations have only first derivatives.

Problem 75. Derive a wave equation for the electric field $\mathbf{E}$ from the free-space Maxwell equations. (Hint: Take the curl of Faraday's law, take the time derivative of the Ampere-Maxwell law, subtract, and simplify.)

Problem 76. Compare this wave equation to (1), (6), and (11), and say what you think the speed of EM waves will be based on this comparison.

Problem 77. Derive a wave equation for the magnetic field $\mathbf{B}$ from the free-space Maxwell equations.

We seek wave solutions to the free-space Maxwell equations. Let us begin by trying solutions with an electric field

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0 \cos(\mathbf{k} \cdot \mathbf{r} - \omega t + \delta). \tag{13}$$

4.2 Constraints

Not everything that looks like an EM wave is going to be an EM wave that satisfies the Maxwell equations. There are some constraints.

Problem 78. Substitute (13) into the free-space Gauss's law and obtain a condition that must be satisfied in order for (13) to be a solution to the free-space Maxwell equations. How would you state this condition in words?

Problem 79. Let's prove a little lemma (Do you know what a lemma is? It's a little helping theorem.) that will help us in two of the problems to come. If

$$\mathbf{A}(\mathbf{r}, t) = \mathbf{A}_0 \cos(\mathbf{k} \cdot \mathbf{r} - \omega t + \delta),$$

show that

$$\nabla \times \mathbf{A} = -\mathbf{k} \times \mathbf{A}_0 \sin(\mathbf{k} \cdot \mathbf{r} - \omega t + \delta).$$

Problem 80. Substitute (13) into Faraday's law and find all functions $\mathbf{B}(\mathbf{r}, t)$ that will satisfy it.

Problem 81. If, in the previous problem, we ignore the constant magnetic field that comes about as a constant of integration, we get a unique magnetic field that satisfies Faraday's law for the electric field (13). Write down that magnetic field in the box below.

$$\boxed{\qquad\qquad\qquad\qquad} \tag{14}$$

Problem 82. Show that the magnetic field (14) satisfies the "no magnetic monopoles" law.

Problem 83. Substitute (13) and (14) into the free-space Ampere-Maxwell law and give any conditions that must be satisfied for them to be a solution.

Problem 84. In light of the previous problems, state precisely the information that must be given to specify a plane traveling EM wave.

Problem 85. Consider the electromagnetic field

$$\begin{aligned}\mathbf{E}(\mathbf{r},t) &= \mathbf{E}_0 \cos(\mathbf{k}_E \cdot \mathbf{r} - \omega_E t + \delta_E)\\ \mathbf{B}(\mathbf{r},t) &= \mathbf{B}_0 \cos(\mathbf{k}_B \cdot \mathbf{r} - \omega_B t + \delta_B).\end{aligned}$$

How many of the parameters

$$E_{0x}, E_{0y}, E_{0z}, B_{0x}, B_{0y}, B_{0z}, k_{Ex}, k_{Ey}, k_{Ez}, k_{Bx}, k_{By}, k_{Bz}, \omega_E, \omega_B, \delta_E, \delta_B$$

may be chosen independently, and how are the remaining parameters found from the ones chosen?

By convention, the direction of $\mathbf{E}_0$ is called the direction of polarization. So, for example, an EM wave polarized in the x direction would have $\mathbf{E}_0 = E_0\hat{\mathbf{i}}$.

Problem 86. Write down the electric and magnetic fields for a monochromatic plane wave of amplitude E_0, angular frequency ω, and phase angle zero that is traveling in the z direction and polarized in the x direction.

Problem 87. Write down the electric and magnetic fields for a monochromatic plane wave of amplitude E_0, angular frequency ω, and phase angle zero that is traveling in the z direction and polarized in the y direction.

Problem 88. Write down the electric and magnetic fields for a monochromatic plane wave of amplitude E_0, angular frequency ω, and phase angle zero that is traveling in the x direction and polarized in the y direction.

Problem 89. Write down the electric and magnetic fields for a monochromatic plane wave of amplitude E_0, angular frequency ω, and phase angle zero that is traveling in the x direction and polarized in the z direction.

Problem 90. Write down the electric and magnetic fields for a monochromatic plane wave of amplitude E_0, angular frequency ω, and phase angle zero that is traveling in the xy plane, making an angle of $\pi/4$ radians with the positive x axis, and polarized in the z direction.

Problem 91. Write down the electric and magnetic fields for a monochromatic plane wave of amplitude E_0, angular frequency ω, and phase angle zero that is traveling in the xy plane, making an angle of $\pi/4$ radians with the positive x axis, and polarized in the $-z$ direction.

Problem 92. Write down the electric and magnetic fields for a monochromatic plane wave of amplitude E_0, wavenumber k (the wavenumber k is the magnitude of the wavevector $\mathbf{k}$, so that $k = |\mathbf{k}|$), and phase angle zero that is traveling in the xy plane, making an angle of $\pi/6$ radians with the positive x axis, and polarized in the z direction.

Any wave can be expressed as a linear combination of traveling waves.

4.3 Energy and Momentum of EM Waves

Read section 8.1 of Griffiths, then try the next two problems.

Problem 93. Find the energy density associated with the EM wave (13) and (14).

Problem 94. Find the Poynting vector associated with the EM wave (13) and (14).

Read section 8.2 of Griffiths, then try the next problem.

Problem 95. Find the momentum density associated with the EM wave (13) and (14).

4.4 Polarization of EM Waves

Consider the EM wave

$$\mathbf{E}_1(\mathbf{r}, t) = E_0 \cos(kz - kct)\hat{\mathbf{i}} \tag{15}$$

$$\mathbf{B}_1(\mathbf{r}, t) = \frac{E_0}{c} \cos(kz - kct)\hat{\mathbf{j}}. \tag{16}$$

Problem 96. Confirm that (15) and (16) are a solution to the Maxwell equations.

This EM wave is said to be *linearly polarized* in the x direction, because the electric field points in the x direction, and it does so at all points in space, and at all times.

Consider the wave

$$\mathbf{E}_2(\mathbf{r}, t) = E_0 \sin(kz - kct)\hat{\mathbf{i}} \tag{17}$$

$$\mathbf{B}_2(\mathbf{r}, t) = \frac{E_0}{c} \sin(kz - kct)\hat{\mathbf{j}}. \tag{18}$$

Problem 97. Write this wave in our standard form (13) using cosines instead of sines.

Problem 98. Consider the wave

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_1(\mathbf{r}, t) + \mathbf{E}_2(\mathbf{r}, t)$$
$$\mathbf{B}(\mathbf{r}, t) = \mathbf{B}_1(\mathbf{r}, t) + \mathbf{B}_2(\mathbf{r}, t).$$

Does this wave have a well-defined direction of propagation? Is this wave linearly polarized? If so, in what direction?

Problem 99. Write the EM wave

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0 \cos(\mathbf{k} \cdot \mathbf{r} - \omega t + \delta)$$
$$\mathbf{B}(\mathbf{r}, t) = \mathbf{B}_0 \cos(\mathbf{k} \cdot \mathbf{r} - \omega t + \delta)$$

as a linear combination of

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0 \cos(\mathbf{k} \cdot \mathbf{r} - \omega t)$$
$$\mathbf{B}(\mathbf{r}, t) = \mathbf{B}_0 \cos(\mathbf{k} \cdot \mathbf{r} - \omega t)$$

and

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0 \sin(\mathbf{k} \cdot \mathbf{r} - \omega t)$$
$$\mathbf{B}(\mathbf{r}, t) = \mathbf{B}_0 \sin(\mathbf{k} \cdot \mathbf{r} - \omega t).$$

Problem 100. Write down the EM wave traveling in the z direction, linearly polarized in the y direction, with wavenumber k, electric field amplitude E_0, and zero phase angle. Call this wave $\mathbf{E}_3$ and $\mathbf{B}_3$.

Traveling waves that are linearly polarized have electric fields that point in the same direction at all points in space and at all times.

Problem 101. Consider the wave

$$\begin{aligned}\mathbf{E}(\mathbf{r},t) &= \mathbf{E}_1(\mathbf{r},t) + \mathbf{E}_3(\mathbf{r},t)\\ \mathbf{B}(\mathbf{r},t) &= \mathbf{B}_1(\mathbf{r},t) + \mathbf{B}_3(\mathbf{r},t).\end{aligned}$$

Does this wave have a well-defined direction of propagation? Is this wave linearly polarized? If so, in what direction?

4.4.1 Circular polarization

Consider the wave

$$\mathbf{E}_4(\mathbf{r},t) = E_0 \sin(kz - kct)\hat{\mathbf{j}} \tag{19}$$

$$\mathbf{B}_4(\mathbf{r},t) = -\frac{E_0}{c} \sin(kz - kct)\hat{\mathbf{i}}. \tag{20}$$

Problem 102. Consider the wave

$$\begin{aligned}\mathbf{E}(\mathbf{r},t) &= \mathbf{E}_1(\mathbf{r},t) + \mathbf{E}_4(\mathbf{r},t)\\ \mathbf{B}(\mathbf{r},t) &= \mathbf{B}_1(\mathbf{r},t) + \mathbf{B}_4(\mathbf{r},t).\end{aligned}$$

Does this wave have a well-defined direction of propagation? Is this wave linearly polarized? If so, in what direction?

Problem 103. For the wave in the previous problem, draw the electric and magnetic field vectors at the origin at time $t = 0$. Draw them also at a slightly later time. What are they doing?

If the electric field at a fixed point in space changes direction in a clockwise way as viewed from a point farther out in the direction of propagation, the wave is called *right circularly polarized*. A *left circularly polarized* wave rotates counterclockwise in time, at a fixed spot, viewed from a point further along the direction of propagation.

Problem 104. Which kind of polarization does the wave of problem 102 have? Construct a wave with the opposite polarization.

Problem 105. Describe the wave that you get if you take a linear combination of wave 1 ($\mathbf{E}_1$ and $\mathbf{B}_1$) and wave 2 with *unequal* coefficients.

Problem 106. Describe the wave that you get if you take a linear combination of wave 1 and wave 3 with unequal coefficients.

Problem 107. Describe the wave that you get if you take a linear combination of wave 1 and wave 4 with unequal coefficients. This wave is said to have *elliptical polarization*. Why is it called that?

5 Modes of the Electromagnetic Field

Every EM wave that travels in the z direction with wavenumber k (so that the wavevector is $\mathbf{k} = k\hat{\mathbf{z}}$) can be expressed as a linear combination of four waves. (Waves 1 through 4 of the last section.)

Now let's start with an arbitrary wavevector $\mathbf{k}$. Let $\mathbf{e}_1$ be a unit vector perpendicular to the wavevector $\mathbf{k}$, and define $\mathbf{e}_2$ to be the unit vector

$$\mathbf{e}_2 = \frac{\mathbf{k}}{|\mathbf{k}|} \times \mathbf{e}_1.$$

We claim that any traveling EM wave with wavevector $\mathbf{k}$ can be written as a linear combination of the following four waves.

$$\mathbf{E}_1(\mathbf{r}, t) = E_0 \cos(\mathbf{k} \cdot \mathbf{r} - kct)\mathbf{e}_1 \tag{21}$$

$$\mathbf{B}_1(\mathbf{r}, t) = \frac{E_0}{c} \cos(\mathbf{k} \cdot \mathbf{r} - kct)\mathbf{e}_2 \tag{22}$$

$$\mathbf{E}_2(\mathbf{r}, t) = E_0 \sin(\mathbf{k} \cdot \mathbf{r} - kct)\mathbf{e}_1 \tag{23}$$

$$\mathbf{B}_2(\mathbf{r}, t) = \frac{E_0}{c} \sin(\mathbf{k} \cdot \mathbf{r} - kct)\mathbf{e}_2 \tag{24}$$

$$\mathbf{E}_3(\mathbf{r}, t) = E_0 \cos(\mathbf{k} \cdot \mathbf{r} - kct)\mathbf{e}_2 \tag{25}$$

$$\mathbf{B}_3(\mathbf{r}, t) = -\frac{E_0}{c} \cos(\mathbf{k} \cdot \mathbf{r} - kct)\mathbf{e}_1 \tag{26}$$

$$\mathbf{E}_4(\mathbf{r}, t) = E_0 \sin(\mathbf{k} \cdot \mathbf{r} - kct)\mathbf{e}_2 \tag{27}$$

$$\mathbf{B}_4(\mathbf{r}, t) = -\frac{E_0}{c} \sin(\mathbf{k} \cdot \mathbf{r} - kct)\mathbf{e}_1 \tag{28}$$

These waves are called *modes* of the EM field. There are many ways to choose a set of modes, since there are many choices for the direction of $\mathbf{e}_1$. There are an infinite number of modes of the EM field. For each wavevector $\mathbf{k}$ (of which there are an infinite number) there are four modes.

The significance of these modes is that *any* EM wave can be written as a linear combination of these modes. In the language of linear algebra, the modes form a basis for the vector space of solutions to the free-space Maxwell equations. (I'm lying just a little bit, because I haven't carefully dealt with the degenerate solutions to the wave equation, like the constant and linear solutions. Presumably, we would need to add in a few degenerate modes to be able to form all of those solutions. I'm not going to worry about that.)

Problem 108. In problem 85, you found the number of parameters needed to describe a linearly polarized plane EM wave. How many parameters are needed to describe an arbitrary plane EM wave? (The adjective "plane" here implies a well-defined wave vector.) What are the parameters needed to describe such a wave?

6 Potentials of EM Waves

Now is a good time to glance at section 1.6 of Griffiths, which we skipped in the fall. Here are two important results from that section.

Theorem 1. *If the curl of a vector field* $\mathbf{F}$ *vanishes everywhere, then* $\mathbf{F}$ *can be expressed as the gradient of a scalar potential* V*.*

$$\nabla \times \mathbf{F} = 0 \Longleftrightarrow \mathbf{F} = -\nabla V$$

(The minus sign is purely conventional.)

Theorem 2. *If the divergence of a vector field* $\mathbf{F}$ *vanishes everywhere, then* $\mathbf{F}$ *can be expressed as the curl of a vector potential* $\mathbf{A}$*.*

$$\nabla \cdot \mathbf{F} = 0 \Longleftrightarrow \mathbf{F} = \nabla \times \mathbf{A}$$

We can apply Theorem 2 to Gauss's law for magnetism ($\nabla \cdot \mathbf{B} = 0$) to assert the existence of a magnetic vector potential $\mathbf{A}$.

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{29}$$

Problem 109. Plug equation (29) into Faraday's law and use Theorem 1 above to derive an equation for $\mathbf{E}$ in terms of the scalar and vector potentials. How is this equation different from the relationship in electrostatics?

Problem 110. Write down scalar and vector potentials that give rise to (13) and (14).