Electromagnetic Theory II

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Chapter 1

Electromagnetic Waves

Recall the Maxwell equations.

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \qquad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$$

$$\nabla \times \mathbf{B} - \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{J} \qquad \nabla \cdot \mathbf{B} = 0$$

Problem 1. Give the names for each of the Maxwell equations.

Problem 2. For each of the following items, give its name, give its units, and classify the item as a scalar field, a vector field, or something else.
(a) $\nabla \cdot$
(b) $\mathbf{E}$
(c) ρ
(d) ϵ_0
(e) $\nabla \times$
(f) $\mathbf{B}$
(g) μ_0
(h) t
(i) $\mathbf{J}$

Our attitude at the moment is to imagine that we know the charge and current distributions, and we want to solve for the electric and magnetic fields. This attitude leads us to think of the Maxwell equations as a set of coupled partial differential equations.

Problem 3. Why are they partial differential equations, and not ordinary differential equations? Why are they coupled?

Problem 4. Write out the Maxwell equations in Cartesian coordinates as a set of coupled partial differential equations. Use the components E_x, E_y, and E_z rather than the vector $\mathbf{E}$. Use the coordinates x, y, and z, and get rid of the ∇ symbol. How many equations are there in all? What are the independent variables? What are the dependent variables? Are the Maxwell equations linear? Homogeneous? First order? Second order?

Problem 5. Are the Maxwell equations "better" than Coulomb's law for describing electricity? If so, how?

Problem 6. Can you name some ways in which electricity and magnetism are related?

In this chapter, we are interested in the properties of electromagnetic (EM) waves. The Maxwell equations can be used to show how accelerating charges and oscillating currents create EM waves. The Maxwell equations can also be used to show how EM waves interact with matter, for example by reflecting, refracting, and scattering. Our goals for this chapter are more fundamental. The most basic thing that a wave can do (more basic than being created, or scattering, or getting absorbed) is to propagate. This chapter is about the *propagation* of EM waves. Notably, the Maxwell equations also describe the propagation of EM waves. The simplest propagation is through empty space, where there is no matter, no charges, and no currents.

For the rest of this chapter, we are interested in solutions to the Maxwell equations in free space, that is, in regions where there are no charges and no currents.

Problem 7. Write down the free-space Maxwell equations.

Problem 8. Find the simplest solution to the free-space Maxwell equations.

Problem 9. Find another solution to the free-space Maxwell equations.

In order for a solution of the free-space Maxwell equations to be regarded as a wave, it needs to oscillate in space and it needs to oscillate in time.

Electromagnetic waves are rather complicated waves, because we have two vector quantities (the electric and magnetic fields) that depend on three space dimensions and a time dimension. Before we search for wave-like solutions to the free-space Maxwell equations, let us digress to think about waves in a simpler setting.

1.1 Scalar Waves

1.1.1 Scalar waves in one space dimension

Let's take a pressure wave as an example of a scalar wave. A pressure wave is a scalar wave because pressure is a scalar quantity. To start, let's make things even simpler and work in a fantasy world that has only one space dimension x and one time dimension t. Our pressure wave is described by a function $P(x,t)$. This function tells you the pressure at position x and time t. The function P will satisfy a wave equation

$$\frac{\partial^2 P}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 P}{\partial t^2} = 0, \tag{1.1}$$

where v is a constant with units of velocity that depends on the properties of the fluid through which the pressure wave propagates. Every function that describes a wave satisfies a partial differential equation like this. (Ordinary differential equations can be used to describe oscillations, but we need partial differential equations to describe waves.)

Problem 10. Find the simplest solution to the wave equation (1.1).

Problem 11. Find another solution to the wave equation (1.1).

In order for a solution of the wave equation (1.1) to be regarded as a nontrivial wave, it needs to oscillate in space and it needs to oscillate in time.

Our prototypical solution to wave equation (1.1) is

$$P(x,t) = P_0 \cos(kx - \omega t), \tag{1.2}$$

where P_0 is a constant with units of pressure, k is called the *wavenumber*, and has units of inverse length, and ω is the angular frequency, with units of radians per unit time.

Notice that $P(x,t)$ can be negative. This is because $P(x,t)$ is a description of the *relative* pressure (also called *gauge* pressure) at position x at time t, and not the absolute pressure. Relative pressure means pressure with respect to the equilibrium pressure of the fluid through which the pressure wave propagates.

Problem 12. What is the wavelength of (1.2)? Explain how you know.

Problem 13. What is the period of (1.2)? Explain how you know.

Problem 14. What is the frequency (denoted f or ν) of (1.2)? Explain how you know.

Points of constant phase

A useful way to think about what is happening with a traveling wave in one spacial dimension, like (1.2), is to examine the *points of constant phase*. These are the places (values of x) at which the pressure is the same. These places will change in time. For example, suppose we decide to follow wave crests. These are the places where the pressure is maximum, $P(x,t) = P_0$. The cosine in (1.2) must give one at these places. Therefore, the argument of the cosine function must be a multiple of 2π.

Problem 15. For the wave described in (1.2), write a relationship between k, x, ω, and t that must hold at wave crests. Use this expression to give the position of a wave crest as a function of time.

Problem 16. What is the velocity of a point on the crest of the wave (1.2)?

We don't need to look at wave crests only. We could choose to follow wave troughs (in which case we would take $P(x,t) = -P_0$), points of equilibrium pressure (in which case we would take $P(x,t) = 0$), or any other points (say $P(x,t) = P_0/2$, for example). In all of these cases, the points of constant phase have the same velocity.

Problem 17. Show that points on the wave trough of (1.2) have the same velocity as points on the wave crest.

Because, for a traveling wave like (1.2), any point of constant phase moves with the same velocity, we will call this the *wave velocity*.

Problem 18. Insert (1.2) into wave equation (1.1), and find a relationship between v, k, and ω that needs to hold in order for (1.2) to be a solution of (1.1). In light of this relationship, what is the meaning of v in (1.1)?

Problem 19. Rewrite the relationship you found in the previous problem in terms of wave speed v, wavelength λ, and period T. Give a physical interpretation for this relationship. (In other words, what does it mean?) Also, give the relationship between v, λ, and ν.

Problem 20. In what direction is wave (1.2) propagating? Explain how you know.

Problem 21. Write an expression for a wave propagating in the opposite direction.

Problem 22. For wave (1.2), sketch a graph of P as a function of x at a fixed time t. Sketch a second graph of P vs. x at a slightly later time, and a third graph at a still later time.

Phase angle

Wave (1.2) has maximum pressure at $x = 0$ at $t = 0$. There are solutions to (1.1) very much like (1.2) that do not have this attribute.

Problem 23. Show that

$$P(x,t) = P_0 \cos(kx - \omega t + \delta) \tag{1.3}$$

is a solution to (1.1) for any real number δ. We call δ a *phase angle*.

Problem 24. Find a phase angle so that (1.3) is a wave with minimum pressure at $x = 0$, $t = 0$.

Problem 25. Find a phase angle so that (1.3) is a wave with zero pressure at $x = 0$, $t = 0$. In fact, there are two different waves that have this feature. See if you can find both.

Standing waves

A standing wave is a wave with nodes. A node is a place where the (relative) pressure is zero and remains zero over time (remember zero relative pressure means the equilibrium pressure of the fluid).

Problem 26. At $t = 0$, a pressure wave has nodes at $-3L$, $-2L$, $-L$, 0, L, $2L$, $3L$, etc. Write a function of position x describing the pressure.

Problem 27. Take the function of position from the previous problem and make it into a wave by multiplying it by $\cos \omega t$. Show that the wave satisfies the wave equation (1.1). Are there conditions that must hold in order for this standing wave to be a solution?

Problem 28. For the standing wave you formed in the previous problem, sketch a graph of P as a function of x at a fixed time t. Sketch a second graph of P vs. x at a slightly later time, and a third graph at a still later time.

Problem 29. What is the wavelength of the standing wave? What is the period of the standing wave?

Problem 30. Do the wavelength, period, and speed v have the same relationship for the standing wave that they do for the traveling wave?

Problem 31. Consider the standing wave

$$P(x,t) = P_0 \sin kx \cos \omega t.$$

Where are the nodes of this standing wave? What condition must hold for this wave to satisfy the wave equation (1.1)? Write this standing wave as a sum of a traveling wave propagating in the positive x direction and a traveling wave propagating in the negative x direction. (You will need to use some trigonometric identities to do this.)

Problem 32. Consider the standing wave

$$P(x,t) = P_0 \sin kx \sin \omega t.$$

Where are the nodes of this standing wave? What condition must hold for this wave to satisfy the wave equation (1.1)? Write this standing wave as a sum of a traveling wave propagating in the positive x direction and a traveling wave propagating in the negative x direction.

Problem 33. Consider the standing wave

$$P(x,t) = P_0 \cos kx \cos \omega t.$$

Where are the nodes of this standing wave? What condition must hold for this wave to satisfy the wave equation (1.1)? Write this standing wave as a sum of a traveling wave propagating in the positive x direction and a traveling wave propagating in the negative x direction.

Problem 34. Consider the standing wave

$$P(x,t) = P_0 \cos kx \sin \omega t.$$

Where are the nodes of this standing wave? What condition must hold for this wave to satisfy the wave equation (1.1)? Write this standing wave as a sum of a traveling wave propagating in the positive x direction and a traveling wave propagating in the negative x direction.

1.1.2 Scalar waves in two space dimensions

Let's continue to consider a pressure wave as an example of a scalar wave. Let's make things slightly richer by working in a world that has two space dimensions x and y, and one time dimension t. Our pressure wave is now described by a function $P(x, y, t)$. This function tells you what the pressure is at position (x, y) and time t. The function P will satisfy a wave equation

$$\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} - \frac{1}{v^2}\frac{\partial^2 P}{\partial t^2} = 0, \tag{1.4}$$

where v is a constant with units of velocity that depends on the properties of the fluid through which the pressure wave propagates.

Our favorite solution to (1.4) is

$$P(x, y, t) = P_0 \cos(k_x x + k_y y - \omega t + \delta). \tag{1.5}$$

Problem 35. Insert (1.5) into (1.4) and find any conditions that must hold in order for (1.5) to be a solution to (1.4).

We wish to understand the meaning of k_x and k_y in (1.5). To help us understand that, we will look at the lines of constant phase.

Lines of constant phase

In two spacial dimensions, a wave like (1.5) has lines of constant phase rather than points of constant phase.

Problem 36. For the wave described in (1.5), write the equation of a line in the xy plane that describes one of the wave crests at time t.

Problem 37. Now write the equation of a line in the xy plane that describes one of the wave crests at time $t + \Delta t$. Sketch the line of wave crests at time t in the xy plane and the line of wave crests at time $t + \Delta t$ on the same picture.

Problem 38. Find the distance between these two lines in the xy plane.

Problem 39. Use the result of the previous problem to find the speed of the wave.

Problem 40. Find the lines of constant phase for the wave (1.5) with $k_x = \frac{1}{\sqrt{2}}\frac{\omega}{v}$, $k_y = \frac{1}{\sqrt{2}}\frac{\omega}{v}$, and $\delta = 0$.

Problem 41. What is the direction of propagation for the wave in the previous problem?

Problem 42. Find the lines of constant phase for the wave (1.5) with $k_x = \frac{\omega}{v}$, $k_y = 0$, and $\delta = 0$. Give the direction of propagation.

Problem 43. Find the lines of constant phase for the wave (1.5) with $k_x = 0$, $k_y = \frac{\omega}{v}$, and $\delta = 0$. Give the direction of propagation.

Problem 44. What can you say about the relationship between the lines of constant phase and the direction of propagation?

Problem 45. What can you say about the relationship between the vector $k_x\hat{\mathbf{i}} + k_y\hat{\mathbf{j}}$ (called the wave vector) and the direction of propagation?

Problem 46. Write a pressure wave in 2D traveling in the negative x direction, with angular frequency ω, amplitude P_0, and wave speed v.

Problem 47. Write a pressure wave in 2D traveling in the negative y direction, with wavelength λ, amplitude P_0, and wave speed v.

A wave is any solution to a wave equation. A traveling wave is a wave with a well-defined direction of propagation.

Problem 48. Give an example of a 2D wave without a well-defined direction of propagation.

Problem 49. Give an example of a 1D wave without a well-defined direction of propagation.

Problem 50. Sketch a 2D traveling wave (P as a function of x and y) at t, $t + \Delta t$, and $t + 2\Delta t$.

Standing waves in 2D

Problem 51. Is

$$P(x, y, t) = P_0 \sin k_x x \sin k_y y \cos \omega t$$

a solution to wave equation (1.4)? If so, describe its nodes.

Problem 52. Is

$$P(x, y, t) = P_0 \sin k_x x \cos(k_y y - \omega t)$$

a solution to wave equation (1.4)? If so, describe its nodes.

Problem 53. Write an expression for a pressure wave $P(x, y, t)$ with amplitude P_0, wave velocity v and the following nodes. The lines $x = 0$ m, $x = 2$ m, $x = -2$ m, $x = 4$ m, $x = -4$ m, and so on are nodes. The lines $y = 0$ m, $y = 3$ m, $y = -3$ m, $y = 6$ m, $y = -6$ m, and so on are nodes.

Linearity

Wave equations (1.1) and (1.4) are *linear*, *homogeneous* partial differential equations. This means that the sum of any two (or more) solutions is also a solution. In fact, we can build up all possible solutions as sums of traveling wave solutions.

1.1.3 Scalar waves in three space dimensions

Let's continue to consider a pressure wave as an example of a scalar wave. We now work in a world that has three space dimensions x, y, and z, and one time dimension t. Our pressure wave is now described by a function $P(x, y, z, t)$. This function tells you what the pressure is at position (x, y, z) and time t.

Problem 54. Write down the wave equation that $P(x, y, z, t)$ will satisfy. Use Cartesian coordinates.

Problem 55. Rewrite the wave equation that $P(\mathbf{r}, t)$ will satisfy, using the Laplacian operator ∇^2. Put it in the box below. This wave equation is now expressed in a coordinate-free manner, because it does not make explicit reference to the Cartesian coordinates x, y, and z.

$$\boxed{\qquad\qquad\qquad\qquad\qquad\qquad} \qquad (1.6)$$

Problem 56. In analogy with (1.3) and (1.5), write down the general traveling wave solution to (1.6) in Cartesian coordinates.

Problem 57. Rewrite the solution in a coordinate-free manner, in terms of the vector $\mathbf{k}$, the vector $\mathbf{r}$, and the dot product. Put it in the box below.

$$\boxed{\qquad\qquad\qquad\qquad} \tag{1.7}$$

Problem 58. Give the conditions that must hold for (1.7) to be a solution to (1.6).

Planes of constant phase

In three spacial dimensions, a wave like (1.7) has planes of constant phase rather than points or lines of constant phase.

Problem 59. Find the planes of constant phase (lets choose wave crests, to be concrete) for the wave (1.7) with

$$\mathbf{k} = \frac{\omega}{v}\left(\frac{1}{\sqrt{2}}\hat{\mathbf{j}} + \frac{1}{\sqrt{2}}\hat{\mathbf{k}}\right)$$

and $\delta = 0$.

Problem 60. What is the direction of propagation for the wave in the previous problem?

Problem 61. What can you say about the relationship between the planes of constant phase and the direction of propagation?

Problem 62. What can you say about the relationship between the vector $\mathbf{k}$ and the direction of propagation?

Problem 63. Write a pressure wave in 3D traveling in the direction $\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}}$ with angular frequency ω, amplitude P_0, and wave speed v.

Problem 64. Write a pressure wave in 3D traveling in the negative y direction, with period T, amplitude P_0, and wave speed v.

1.2 EM Waves

1.2.1 EM Wave Equation

The Maxwell equations do not look like a wave equation. A wave equation is a partial differential equation that has second derivatives in space and a second derivative in time. The Maxwell equations have only first derivatives.

Problem 65. Derive a wave equation for the electric field $\mathbf{E}$ from the free-space Maxwell equations. (Hint: Take the curl of Faraday's law, take the time derivative of the Ampere-Maxwell law, subtract, and simplify.)

Problem 66. Compare this wave equation to (1.1), (1.4), and (1.6), and say what you think the speed of EM waves will be based on this comparison.

Problem 67. Derive a wave equation for the magnetic field $\mathbf{B}$ from the free-space Maxwell equations.

We seek wave solutions to the free-space Maxwell equations. Let us begin by trying solutions with an electric field

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0 \cos(\mathbf{k} \cdot \mathbf{r} - \omega t + \delta). \tag{1.8}$$

1.2.2 Constraints

Not everything that looks like an EM wave is going to be an EM wave that satisfies the Maxwell equations. There are some constraints.

Problem 68. Substitute (1.8) into the free-space Gauss's law and obtain a condition that must be satisfied in order for (1.8) to be a solution to the free-space Maxwell equations. How would you state this condition in words?

Problem 69. Let's prove a little lemma (Do you know what a lemma is? It's a little helping theorem.) that will help us in two of the problems to come. If

$$\mathbf{A}(\mathbf{r}, t) = \mathbf{A}_0 \cos(\mathbf{k} \cdot \mathbf{r} - \omega t + \delta),$$

show that

$$\nabla \times \mathbf{A} = -\mathbf{k} \times \mathbf{A}_0 \sin(\mathbf{k} \cdot \mathbf{r} - \omega t + \delta).$$

Problem 70. Substitute (1.8) into Faraday's law and find all functions $\mathbf{B}(\mathbf{r}, t)$ that will satisfy it.

Problem 71. If, in the previous problem, we ignore the constant magnetic field that comes about as a constant of integration, we get a unique magnetic field that satisfies Faraday's law for the electric field (1.8). Write down that magnetic field in the box below.

$$\boxed{} \tag{1.9}$$

Problem 72. Show that the magnetic field (1.9) satisfies the "no magnetic monopoles" law.

Problem 73. Substitute (1.8) and (1.9) into the free-space Ampere-Maxwell law and give any conditions that must be satisfied for them to be a solution.

Problem 74. In light of the previous problems, state precisely the information that must be given to specify a plane traveling EM wave.

Problem 75. Consider the electromagnetic field

$$\begin{aligned}\mathbf{E}(\mathbf{r},t) &= \mathbf{E}_0\cos(\mathbf{k}_E\cdot\mathbf{r} - \omega_E t + \delta_E)\\ \mathbf{B}(\mathbf{r},t) &= \mathbf{B}_0\cos(\mathbf{k}_B\cdot\mathbf{r} - \omega_B t + \delta_B).\end{aligned}$$

How many of the parameters

$$E_{0x}, E_{0y}, E_{0z}, B_{0x}, B_{0y}, B_{0z}, k_{Ex}, k_{Ey}, k_{Ez}, k_{Bx}, k_{By}, k_{Bz}, \omega_E, \omega_B, \delta_E, \delta_B$$

may be chosen independently, and how are the remaining parameters found from the ones chosen?

By convention, the direction of $\mathbf{E}_0$ is called the direction of polarization. So, for example, an EM wave polarized in the x direction would have $\mathbf{E}_0 = E_0\hat{\mathbf{i}}$.

Problem 76. Write down the electric and magnetic fields for a monochromatic plane wave of amplitude E_0, angular frequency ω, and phase angle zero that is traveling in the z direction and polarized in the x direction.

Problem 77. Write down the electric and magnetic fields for a monochromatic plane wave of amplitude E_0, angular frequency ω, and phase angle zero that is traveling in the z direction and polarized in the y direction.

Problem 78. Write down the electric and magnetic fields for a monochromatic plane wave of amplitude E_0, angular frequency ω, and phase angle zero that is traveling in the x direction and polarized in the y direction.

Problem 79. Write down the electric and magnetic fields for a monochromatic plane wave of amplitude E_0, angular frequency ω, and phase angle zero that is traveling in the x direction and polarized in the z direction.

Problem 80. Write down the electric and magnetic fields for a monochromatic plane wave of amplitude E_0, angular frequency ω, and phase angle zero that is traveling in the xy plane, making an angle of $\pi/4$ radians with the positive x axis, and polarized in the z direction.

Problem 81. Write down the electric and magnetic fields for a monochromatic plane wave of amplitude E_0, angular frequency ω, and phase angle zero that is traveling in the xy plane, making an angle of $\pi/4$ radians with the positive x axis, and polarized in the $-z$ direction.

Problem 82. Write down the electric and magnetic fields for a monochromatic plane wave of amplitude E_0, wavenumber k (the wavenumber k is the magnitude of the wavevector $\mathbf{k}$, so that $k = |\mathbf{k}|$), and phase angle zero that is traveling in the xy plane, making an angle of $\pi/6$ radians with the positive x axis, and polarized in the z direction.

Any wave can be expressed as a linear combination of traveling waves.

1.2.3 Energy and Momentum of EM Waves

Read section 8.1 of Griffiths, then try the next two problems.

Problem 83. Find the energy density associated with the EM wave (1.8) and (1.9).

Problem 84. Find the Poynting vector associated with the EM wave (1.8) and (1.9).

Read section 8.2 of Griffiths, then try the next problem.

Problem 85. Find the momentum density associated with the EM wave (1.8) and (1.9).

1.2.4 Polarization of EM Waves

Consider the EM wave

$$\mathbf{E}_1(\mathbf{r},t) = E_0 \cos(kz - kct)\hat{\mathbf{i}} \tag{1.10}$$

$$\mathbf{B}_1(\mathbf{r},t) = \frac{E_0}{c} \cos(kz - kct)\hat{\mathbf{j}}. \tag{1.11}$$

Problem 86. Confirm that (1.10) and (1.11) are a solution to the Maxwell equations.

This EM wave is said to be *linearly polarized* in the x direction, because the electric field points in the x direction, and it does so at all points in space, and at all times.

Consider the wave

$$\mathbf{E}_2(\mathbf{r},t) = E_0 \sin(kz - kct)\hat{\mathbf{i}} \tag{1.12}$$

$$\mathbf{B}_2(\mathbf{r},t) = \frac{E_0}{c} \sin(kz - kct)\hat{\mathbf{j}}. \tag{1.13}$$

Problem 87. Write this wave in our standard form (1.8) using cosines instead of sines.

Problem 88. Consider the wave

$$\mathbf{E}(\mathbf{r},t) = \mathbf{E}_1(\mathbf{r},t) + \mathbf{E}_2(\mathbf{r},t)$$
$$\mathbf{B}(\mathbf{r},t) = \mathbf{B}_1(\mathbf{r},t) + \mathbf{B}_2(\mathbf{r},t).$$

Does this wave have a well-defined direction of propagation? Is this wave linearly polarized? If so, in what direction?

Problem 89. Write the EM wave

$$\mathbf{E}(\mathbf{r},t) = \mathbf{E}_0 \cos(\mathbf{k}\cdot\mathbf{r} - \omega t + \delta)$$
$$\mathbf{B}(\mathbf{r},t) = \mathbf{B}_0 \cos(\mathbf{k}\cdot\mathbf{r} - \omega t + \delta)$$

as a linear combination of

$$\mathbf{E}(\mathbf{r},t) = \mathbf{E}_0 \cos(\mathbf{k}\cdot\mathbf{r} - \omega t)$$
$$\mathbf{B}(\mathbf{r},t) = \mathbf{B}_0 \cos(\mathbf{k}\cdot\mathbf{r} - \omega t)$$

and

$$\mathbf{E}(\mathbf{r},t) = \mathbf{E}_0 \sin(\mathbf{k}\cdot\mathbf{r} - \omega t)$$
$$\mathbf{B}(\mathbf{r},t) = \mathbf{B}_0 \sin(\mathbf{k}\cdot\mathbf{r} - \omega t).$$

Problem 90. Write down the EM wave traveling in the z direction, linearly polarized in the y direction, with wavenumber k, electric field amplitude E_0, and zero phase angle. Call this wave $\mathbf{E}_3$ and $\mathbf{B}_3$.

Traveling waves that are linearly polarized have electric fields that point in the same direction at all points in space and at all times.

Problem 91. Consider the wave

$$\mathbf{E}(\mathbf{r},t) = \mathbf{E}_1(\mathbf{r},t) + \mathbf{E}_3(\mathbf{r},t)$$
$$\mathbf{B}(\mathbf{r},t) = \mathbf{B}_1(\mathbf{r},t) + \mathbf{B}_3(\mathbf{r},t).$$

Does this wave have a well-defined direction of propagation? Is this wave linearly polarized? If so, in what direction?

Circular polarization

Consider the wave

$$\mathbf{E}_4(\mathbf{r},t) = E_0 \sin(kz - kct)\hat{\mathbf{j}} \tag{1.14}$$
$$\mathbf{B}_4(\mathbf{r},t) = -\frac{E_0}{c} \sin(kz - kct)\hat{\mathbf{i}}. \tag{1.15}$$

Problem 92. Consider the wave

$$\mathbf{E}(\mathbf{r},t) = \mathbf{E}_1(\mathbf{r},t) + \mathbf{E}_4(\mathbf{r},t)$$
$$\mathbf{B}(\mathbf{r},t) = \mathbf{B}_1(\mathbf{r},t) + \mathbf{B}_4(\mathbf{r},t).$$

Does this wave have a well-defined direction of propagation? Is this wave linearly polarized? If so, in what direction?

Problem 93. For the wave in the previous problem, draw the electric and magnetic field vectors at the origin at time $t = 0$. Draw them also at a slightly later time. What are they doing?

If the electric field at a fixed point in space changes direction in a clockwise way as viewed from a point farther out in the direction of propagation, the wave is called *right circularly polarized*. A *left circularly polarized* wave rotates counterclockwise in time, at a fixed spot, viewed from a point further along the direction of propagation.

Problem 94. Which kind of polarization does the wave of problem 92 have? Construct a wave with the opposite polarization.

Problem 95. Describe the wave that you get if you take a linear combination of wave 1 ($\mathbf{E}_1$ and $\mathbf{B}_1$) and wave 2 with *unequal* coefficients.

Problem 96. Describe the wave that you get if you take a linear combination of wave 1 and wave 3 with unequal coefficients.

Problem 97. Describe the wave that you get if you take a linear combination of wave 1 and wave 4 with unequal coefficients. This wave is said to have *elliptical polarization*. Why is it called that?

1.3 Modes of the Electromagnetic Field

Every EM wave that travels in the z direction with wavenumber k (so that the wavevector is $\mathbf{k} = k\hat{\mathbf{z}}$) can be expressed as a linear combination of four waves. (Waves 1 through 4 of the last section.)

Now let's start with an arbitrary wavevector $\mathbf{k}$. Let $\mathbf{e}_1$ be a unit vector perpendicular to the wavevector $\mathbf{k}$, and define $\mathbf{e}_2$ to be the unit vector

$$\mathbf{e}_2 = \frac{\mathbf{k}}{|\mathbf{k}|} \times \mathbf{e}_1.$$

We claim that any traveling EM wave with wavevector $\mathbf{k}$ can be written as a linear combination of the following four waves.

$$\mathbf{E}_1(\mathbf{r},t) = E_0 \cos(\mathbf{k}\cdot\mathbf{r} - kct)\mathbf{e}_1 \tag{1.16}$$

$$\mathbf{B}_1(\mathbf{r},t) = \frac{E_0}{c} \cos(\mathbf{k}\cdot\mathbf{r} - kct)\mathbf{e}_2 \tag{1.17}$$

$$\mathbf{E}_2(\mathbf{r},t) = E_0 \sin(\mathbf{k}\cdot\mathbf{r} - kct)\mathbf{e}_1 \tag{1.18}$$

$$\mathbf{B}_2(\mathbf{r},t) = \frac{E_0}{c} \sin(\mathbf{k}\cdot\mathbf{r} - kct)\mathbf{e}_2 \tag{1.19}$$

$$\mathbf{E}_3(\mathbf{r},t) = E_0 \cos(\mathbf{k}\cdot\mathbf{r} - kct)\mathbf{e}_2 \tag{1.20}$$

$$\mathbf{B}_3(\mathbf{r},t) = -\frac{E_0}{c} \cos(\mathbf{k}\cdot\mathbf{r} - kct)\mathbf{e}_1 \tag{1.21}$$

$$\mathbf{E}_4(\mathbf{r},t) = E_0 \sin(\mathbf{k}\cdot\mathbf{r} - kct)\mathbf{e}_2 \tag{1.22}$$

$$\mathbf{B}_4(\mathbf{r},t) = -\frac{E_0}{c} \sin(\mathbf{k}\cdot\mathbf{r} - kct)\mathbf{e}_1 \tag{1.23}$$

These waves are called *modes* of the EM field. There are many ways to choose a set of modes, since there are many choices for the direction of $\mathbf{e}_1$. There are an infinite number of modes of the EM field. For each wavevector $\mathbf{k}$ (of which there are an infinite number) there are four modes.

The significance of these modes is that *any* EM wave can be written as a linear combination of these modes. In the language of linear algebra, the modes form a basis for the vector space of solutions to the free-space Maxwell equations. (I'm lying just a little bit, because I haven't carefully dealt with the degenerate solutions to the wave equation, like the constant and linear solutions. Presumably, we would need to add in a few degenerate modes to be able to form all of those solutions. I'm not going to worry about that.)

Problem 98. In problem 75, you found the number of parameters needed to describe a linearly polarized plane EM wave. How many parameters are needed to describe an arbitrary plane EM wave? (The adjective "plane" here implies a well-defined wave vector.) What are the parameters needed to describe such a wave?

1.4 Potentials of EM Waves

Now is a good time to glance at section 1.6 of Griffiths, which we skipped in the fall. Here are two important results from that section.

Theorem 1. *If the curl of a vector field* $\mathbf{F}$ *vanishes everywhere, then* $\mathbf{F}$ *can be expressed as the gradient of a scalar potential* V*.*

$$\nabla \times \mathbf{F} = 0 \Longleftrightarrow \mathbf{F} = -\nabla V$$

(The minus sign is purely conventional.)

Theorem 2. *If the divergence of a vector field* $\mathbf{F}$ *vanishes everywhere, then* $\mathbf{F}$ *can be expressed as the curl of a vector potential* $\mathbf{A}$*.*

$$\nabla \cdot \mathbf{F} = 0 \Longleftrightarrow \mathbf{F} = \nabla \times \mathbf{A}$$

We can apply Theorem 2 to Gauss's law for magnetism ($\nabla \cdot \mathbf{B} = 0$) to assert the existence of a magnetic vector potential $\mathbf{A}$.

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{1.24}$$

Problem 99. Plug equation (1.24) into Faraday's law and use Theorem 1 above to derive an equation for $\mathbf{E}$ in terms of the scalar and vector potentials. How is this equation different from the relationship in electrostatics?

Problem 100. Write down scalar and vector potentials that give rise to (1.8) and (1.9).

Chapter 2

EM Fields in Matter

2.1 The Fields D and H

2.1.1 Polarization

Matter is made of protons and electrons, and so when matter is exposed to an electric field, the positive and negative charges in the matter respond to the electric field.

Some materials, like metals, contain electrons that are essentially free to move around in the material. These materials are called *conductors*. The electrons in a conductor respond to an electric field by redistributing, in such a way that they create their own electric field that cancels the applied electric field. An *ideal conductor* is one in which the electric field is always zero, because electrons can respond perfectly to any applied electric field.

Our main concern in this chapter is not with conductors, but with materials that are called *dielectrics*. In a dielectric, the electrons are not free to move around the material in response to an applied electric field, but are bound to a particular atom or molecule. Nevertheless, these electrons can move a little bit in response to an applied electric field, and they will. Because the electrons in a dielectric are bound to particular atoms or molecules, dielectrics are electrical insulators.

When a dielectric is exposed to an electric field, it becomes polarized. This means that the positive charges tend to move a bit in the direction of the electric field and the negative charges tend to move a bit in the opposite direction. Each molecule in the dielectric develops a dipole moment in response to the applied electric field. We define a polarization vector $\mathbf{P}$ to be the electric dipole moment per unit volume. The polarization (also called *dielectric polarization*) is larger for applied electric fields that are larger. For a *linear material*, the polarization is directly proportional to the applied electric field. For applied electric fields that are not too high, most dielectric materials behave as linear materials.

Note that we are using the term *polarization* differently than we did in the last chapter. In the last chapter, we spoke about the polarization of EM waves by labeling them linearly polarized or circularly polarized or elliptically polarized. Our use of the word polarization in this chapter is essentially different and distinct from that in Chapter 1. In this chapter, the polarization $\mathbf{P}$ describes the matter, not the EM field. Here, polarization is dipole moment per unit volume.

Problem 101. What are the units for dipole moment? What are the units for the polarization $\mathbf{P}$?

The electric susceptibility χ_e of a linear material describes how large the polarization is for a given applied electric field.

$$\mathbf{P} = \chi_e \mathbf{E}$$

Because of the polarization, the electric field inside the material is not the same as the applied electric field. The electric field inside the material is the sum of the external (applied) electric field and the internal electric field generated by the polarization because of the new positions of the charges.

How much electric field is generated by this polarization? This question is difficult to answer, and it is the subject of section 4.2.1 in Griffiths' book, if you'd like to read about it. The bottom line is that the polarization creates an electric field just like that created by an electric charge density

$$\rho_b(\mathbf{r}) = -\nabla \cdot \mathbf{P}(\mathbf{r}).$$

The subscript "b" in ρ_b stands for "bound," because this charge constitutes the matter that makes up the dielectric, and it is not free to move around as it pleases.

In fact we don't really want to deal with this bound charge at all. The Maxwell equations tell us that we need to be concerned with the charge density ρ. If we have matter around (which we generally do, the darn stuff is everywhere) and matter is made up of particles with electric charge, then in principle we need to keep track of all that charge. We really don't want to do that. We want to get out of that job. We're looking for a system in which we only keep track of the free charge, which is charge that's not contained in a slab of dielectric material.

We are going to divide the total charge into two parts, a free part and a bound part.

$$\rho(\mathbf{r}) = \rho_f(\mathbf{r}) + \rho_b(\mathbf{r})$$

Let us look at Gauss's law.

$$\begin{aligned}\nabla \cdot \mathbf{E} &= \rho/\epsilon_0 \\ &= \rho_f/\epsilon_0 + \rho_b/\epsilon_0 \\ &= \rho_f/\epsilon_0 - \frac{1}{\epsilon_0}\nabla \cdot \mathbf{P}\end{aligned}$$

Multiplying through by ϵ_0 and rearranging, we have

$$\nabla \cdot (\epsilon_0 \mathbf{E} + \mathbf{P}) = \rho_f.$$

2.1.2 Electric Displacement D

Now we define a new quantity, the electric displacement **D**.

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}$$

In a dielectric, we have a macroscopic version of Gauss's law.

$$\nabla \cdot \mathbf{D} = \rho_f$$

We'll call this "Gauss's law in matter."

What is the status of this new law? Is it an improved version of Gauss's law? Not really. In one small way we could answer yes to this question, but we made a number of assumptions and approximations in deriving it.

Under what conditions should we use this new law? We should use this law when we have a linear material and we don't want to be bothered about keeping track of the charge density of the protons and electrons that make up the material. The only charge density that we worry about in this setting is the free charge density, that is any charge that is not part of what makes up the dielectric material. We should also realize that the field **D** described by Gauss's law in matter is macroscopic average field. It has nothing to tell us about how the electric field might vary as we go from one atom in the material to the next. We just want to think of the material as a slab, not worrying about atoms, and bound charges, and stuff like that.

The *permittivity* ϵ of a material is defined by the relationship

$$\mathbf{D} = \epsilon\mathbf{E}.$$

We can view free space as a dielectric material with permittivity ϵ_0.

Problem 102. Derive a relationship between ϵ and χ_e.

Problem 103. Explain why the permittivity of a linear material must be greater than or equal to ϵ_0. What would it mean physically for a material to have a permittivity of $\epsilon = \epsilon_0$?

The *dielectric constant* K_e is defined to be the dimensionless ratio

$$K_e = \frac{\epsilon}{\epsilon_0}.$$

Problem 104. Using the divergence theorem, derive an integral form of Gauss's law in matter.

Problem 105. Consider a parallel plate capacitor filled with dielectric material with dielectric constant K_e. If the plates have free surface charge densities σ_0 and $-\sigma_0$, find $\mathbf{D}$ and $\mathbf{E}$ inside the capacitor. You may assume that $\mathbf{E} = 0$ outside the capacitor.

Problem 106. Begin with Gauss's law in matter, and take for your "matter" a slab of empty space. Derive the original Gauss's law from this. This is the one sense in which we might regard Gauss's law in matter as a generalization (and hence a "better version") of Gauss's law.

Problem 107. Consider a point charge q at the center of a dielectric sphere of radius R and permittivity ϵ. Regard the point charge as free charge. Find the electric field inside and outside of the dielectric sphere.

2.1.3 Magnetization

Matter is made of protons and electrons, and so when matter is exposed to a magnetic field, the positive and negative charges in the matter respond to the magnetic field.

How do they respond? Well, we know that a moving charged particle feels a force from a magnetic field. We also know that the proton and the electron have intrinsic magnetic dipole moments, so they will feel a torque in an applied magnetic field.

Some materials, like permanent magnets, can have a magnetic dipole moment even in the absence of an applied magnetic field. These are very interesting materials, but we will not talk about them here.

Our main concern in this chapter is with materials that respond to an applied magnetic field by developing a magnetic dipole moment. Materials that do this are called *paramagnetic* if the induced magnetic dipole moment is in the same direction as the applied magnetic field, and *diamagnetic* if the induced magnetic dipole moment is in the opposite direction from the applied magnetic field.

We define a magnetization vector $\mathbf{M}$ to be the magnetic dipole moment per unit volume. The magnetic field in a material is the sum of the applied magnetic field and the magnetic field induced by the magnetization $\mathbf{M}$. How much magnetic field is generated by this magnetization? This question is difficult to answer, and it is the subject of section 6.2.1 of Griffiths' book, if you'd like to read about it. The bottom line is that the magnetization creates a magnetic field just like that created by an electric current density $\nabla \times \mathbf{M}$. This current is a kind of bound current, because it is formed by little current loops of the electrons and protons that make up the matter.

There is another source of bound current as well. A time-varying polarization can create a bound current density that is similar to Maxwell's displacement current. Without going into the details (please see section 6.2.1 of Griffiths' book for more insight), the bound current density can be written

$$\mathbf{J}_b = \nabla \times \mathbf{M} + \frac{\partial \mathbf{P}}{\partial t}.$$

We don't really want to deal with this bound current density at all. The Maxwell equations tell us that we need to be concerned with the current density $\mathbf{J}$. If we have matter around and matter is made up of particles with electric charge, and that electric charge is moving, then in principle we need to keep track of all that current. We really don't want to do that. We want to get out of that job. We're looking for a system in which we only keep track of the free current, which is current that's not simply a response of a material to an applied magnetic field.

We divide the current into two parts, a free part and a bound part.

$$\mathbf{J} = \mathbf{J}_f + \mathbf{J}_b$$

Let us look at the Ampere-Maxwell law.

$$\begin{aligned}\nabla \times \mathbf{B} - \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} &= \mu_0 \mathbf{J} \\ &= \mu_0 \mathbf{J}_f + \mu_0 \nabla \times \mathbf{M} + \mu_0 \frac{\partial \mathbf{P}}{\partial t}\end{aligned}$$

Dividing through by μ_0 and rearranging, we have

$$\nabla \times \left(\frac{1}{\mu_0} \mathbf{B} - \mathbf{M} \right) - \frac{\partial}{\partial t} (\epsilon_0 \mathbf{E} + \mathbf{P}) = \mathbf{J}_f.$$

2.1.4 H

Now we define a new quantity, the "magnetic field intensity" **H**. Many verbal descriptions of **H** have been created over the years. Many people don't like any of the verbal descriptions that have been proposed. Most people just call this field **H**.

$$\mathbf{H} = \frac{1}{\mu_0} \mathbf{B} - \mathbf{M}$$

In matter, we have a macroscopic version of the Ampere-Maxwell law.

$$\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J}_f$$

We'll call this the "Ampere-Maxwell law in matter."

For a *linear material*, the magnetization is directly proportional to the applied magnetic field, and also to **H**. The magnetic susceptibility χ_m of a linear material describes how large the magnetization is for a given **H**.

$$\mathbf{M} = \chi_m \mathbf{H}$$

The *permeability* μ of a material is defined by the relationship

$$\mathbf{B} = \mu \mathbf{H}.$$

We can view free space as a material with permeability μ_0.

Problem 108. Derive a relationship between μ and χ_m.

Problem 109. Using Stokes' theorem, derive an integral form of the Ampere-Maxwell law in matter.

Problem 110. Consider a solenoid filled with material with permeability μ. Suppose the solenoid has n turns of wire per unit length, with a current I flowing through the wire. Find $\mathbf{B}$ and $\mathbf{H}$ inside the solenoid. You may assume that $\mathbf{B} = 0$ outside the solenoid.

Problem 111. Begin with the Ampere-Maxwell law in matter, and take for your "matter" a slab of empty space. Derive the original Ampere-Maxwell law from this.

Problem 112. Consider a long wire carrying current I at the center of a long cylinder of radius R and permeability μ. Regard the current as free current. Find the magnetic field inside and outside of the cylinder.

2.2 The Maxwell Equations in Matter

The Maxwell equations in matter are as follows.

$$\nabla \cdot \mathbf{D} = \rho_f \qquad\qquad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$$

$$\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J}_f \qquad\qquad \nabla \cdot \mathbf{B} = 0$$

Problem 113. Consider a large expanse of material with permittivity ϵ and permeability μ, but with no free charge in it. Electromagnetic waves can propagate in this material. Derive a wave equation for the electric field $\mathbf{E}$ from the Maxwell equations in matter. Looking at this wave equation, what will be different about these EM waves compared with EM waves in free space?

Problem 114. Consider again a material with permittivity ϵ and permeability μ, and no free charge in it. Write expressions for the electric and magnetic fields for the four modes of EM waves with wavevector $\mathbf{k}$ in this material.

2.3 Boundary Conditions

What happens at the interface between two materials with different permittivities and permeabilities? The Maxwell equations, both original recipe and extra crispy (in matter), can help us answer this question.

Let's consider a flat planar interface between two materials. (One of the materials could be vacuum.) We are interested in what happens to **E**, **D**, **B**, and **H** at the interface. In particular, we want to know if these quantities will be continuous across the interface. In other words, will the electric field **E**, for example, have essentially the same value on one side of the interface that it has just micrometers away on the other side?

Problem 115. Consider a planar interface between two materials. The first material has permittivity ϵ_1 and permeability μ_1. The second material has permittivity ϵ_2 and permeability μ_2. Show that if the electric field is not zero and $\epsilon_1 \neq \epsilon_2$, then **D** and **E** cannot both be continuous across the boundary.

The previous problem shows that some of our physical quantities are going to have discontinuities across a boundary between two materials. The question is which ones. The answer, as usual, is contained in the Maxwell equations.

Problem 116. Write down the integral form of each of the Maxwell equations in matter.

For the next several problems, let us consider the following setup. We have one material with permittivity ϵ_1 and permeability μ_1 filling the entire region of space $z < 0$. We have a second material with permittivity ϵ_2 and permeability μ_2 filling the region $z > 0$. The plane $z = 0$ forms the boundary between the two media.

Problem 117. Consider the integral form of Gauss's law in matter. For a Gaussian surface, choose a box with length L in the x direction, width L in the y direction, and height δ in the z direction. The box is centered on the interface, with the top of the box at $z = \delta/2$, and the bottom of the box at $z = -\delta/2$. Assume the box is small enough that the field quantities **E**, **B**, **D**, and **H** don't change significantly from one location to another *within a single material*. Call the field quantities in material 1 $\mathbf{E}_1$, $\mathbf{B}_1$, $\mathbf{D}_1$, and $\mathbf{H}_1$ and those in material 2 $\mathbf{E}_2$, $\mathbf{B}_2$, $\mathbf{D}_2$, and $\mathbf{H}_2$. Write out the integral form of Gauss's law in matter using the Cartesian components of **D**. Then take the limit as $\delta \to 0$. Interpret the result.

Problem 118. Consider the integral form of the "no magnetic monopoles" law in matter. (Also called Gauss's law for magnetism in matter.) For a Gaussian surface, choose a box with length L in the x direction, width L in the y direction, and height δ in the z direction. The box is centered on the interface, with the top of the box at $z = \delta/2$, and the bottom of the box at $z = -\delta/2$. Write out this law using the Cartesian components of $\mathbf{B}$. Then take the limit as $\delta \to 0$. Interpret the result.

Problem 119. Consider the integral form of the Ampere-Maxwell law in matter. For an Amperian loop, choose a rectangle with width L in the y direction, and height δ in the z direction. The rectangle is centered on the interface, with the top of the rectangle at $z = \delta/2$, and the bottom of the rectangle at $z = -\delta/2$. Write this law using the Cartesian components of the field quantities. Then take the limit as $\delta \to 0$. Interpret the result.

Problem 120. Consider the integral form of Faraday's law in matter. For an Amperian loop, choose a rectangle with width L in the y direction, and height δ in the z direction. The rectangle is centered on the interface, with the top of the rectangle at $z = \delta/2$, and the bottom of the rectangle at $z = -\delta/2$. Write this law using the Cartesian components of the field quantities. Then take the limit as $\delta \to 0$. Interpret the result.

There are a couple more boundary conditions that we can derive, this time with the help of the *original* Maxwell equations, which still hold in matter.

Problem 121. Consider the integral form of Gauss's law (original form). For a Gaussian surface, choose a box with length L in the x direction, width L in the y direction, and height δ in the z direction. The box is centered on the interface, with the top of the box at $z = \delta/2$, and the bottom of the box at $z = -\delta/2$. Assume the box is small enough that the field quantities $\mathbf{E}$ and $\mathbf{B}$ don't change significantly from one location to another *within a single material.* Call the field quantities in material 1 $\mathbf{E}_1$ and $\mathbf{B}_1$ and those in material 2 $\mathbf{E}_2$ and $\mathbf{B}_2$. Write out the integral form of Gauss's law (original form) using the Cartesian components of $\mathbf{E}$. Then take the limit as $\delta \to 0$. Interpret the result.

Problem 122. Consider the integral form of the Ampere-Maxwell law (original version). For an Amperian loop, choose a rectangle with width L in the y direction, and height δ in the z direction. The rectangle is centered on the interface, with the top of the rectangle at $z = \delta/2$, and the bottom of the rectangle at $z = -\delta/2$. Write this law using the Cartesian components of the field quantities. Then take the limit as $\delta \to 0$. Interpret the result.

Problem 123. Summarize and organize the findings of the previous six problems by saying in words what happens to the normal component (the component perpendicular to the interface) and the tangential components (the components parallel to the interface) of each of the four field quantities.

Problem 124. Write the nicest looking equations that you can for these boundary conditions. Try to make them "coordinate independent" by avoiding explicit reference to Cartesian components.

Problem 125. (This problem due to Griffiths) The space between the plates of a parallel-plate capacitor is filled with two slabs of linear dielectric material. Each slab has thickness a, so the distance between the plates is $2a$. Slab 1 (next to the top plate) has a dielectric constant of 2, and slab 2 (next to the bottom plate) has a dielectric constant of 1.5. The free charge density on the top plate is σ and on the bottom plate is $-\sigma$.

(a) Find the electric displacement $\mathbf{D}$ in each slab.

(b) Find the electric field $\mathbf{E}$ in each slab.

(c) Find the polarization $\mathbf{P}$ in each slab.

(d) Find the potential difference between the plates.

(e) Find the location and amount of all bound charge.

(f) Now that you know all the charge (free and bound), recalculate the field in each slab, and confirm your answer to (b).

Problem 126. (This problem due to Griffiths) A sphere of linear dielectric material has embedded in it a uniform free charge density ρ. Find the potential at the center of the sphere (relative to infinity), if its radius is R and its dielectric constant is K_e.

Chapter 3

Optics

Optics is an old subject and a very large subject. For as long as people have been aware of light, they have looked for some sort of theory of how light works. I would say that today, there are four active theories of light.

1. Light is a ray.
2. Light is a wave.
3. Light is an electromagnetic wave.
4. Light is a quantum field.

These four theories of light are ordered chronologically, in terms of when they were discovered, and they are also ordered from simplest, most easy to apply, with least predictive power, to most complex, most difficult to apply, with most predictive power.

The achievement of Maxwell is theory 3 of light above, that light is an EM wave. This theory is our primary interest here. Even so, the theory has a lot to say about light and its properties, and we can only cover a small piece here.

Our main interest in this chapter is what happens to a plane wave of light that strikes a plane boundary between two media. Let's work with the following setup. We have one material (let's call it material 1) with permittivity ϵ_1 and permeability μ_1 filling the entire region of space $z < 0$. We have a second material (material 2) with permittivity ϵ_2 and permeability μ_2 filling the region $z > 0$. The plane $z = 0$ forms the boundary between the two media.

We suppose there is a plane EM wave traveling in material 1 toward the boundary with material 2. We call an EM wave that is propagating toward an interface between two materials an *incident wave.* At the interface, some of the energy and momentum of the incident wave is reflected back into material 1, away from the interface, in a wave that we call the *reflected wave.* Another portion of the energy and momentum is transmitted into material 2 in a *transmitted wave* or refracted wave.

We will derive Snell's law as well as expressions for the amount of light reflected and refracted.

3.1 Normal Incidence

First, we investigate what happens when the incident wave is traveling normal to the interface plane, that is, in the z direction. Let us choose the following expressions for our waves.

$$\begin{aligned} \mathbf{E}_i(\mathbf{r},t) &= E_{i0}\cos(k_1 z - \omega t)\hat{\mathbf{i}} \\ \mathbf{E}_r(\mathbf{r},t) &= E_{r0}\cos(-k_1 z - \omega t)\hat{\mathbf{i}} \\ \mathbf{E}_t(\mathbf{r},t) &= E_{t0}\cos(k_2 z - \omega t)\hat{\mathbf{i}} \end{aligned}$$

Problem 127. Write down the corresponding magnetic fields for the incident, reflected, and transmitted waves.

Problem 128. Apply the boundary condition for the tangential component of $\mathbf{E}$ to this situation. Since the boundary is located at $z = 0$, you may substitute $z = 0$ into the waves when you write out the boundary condition. The boundary condition must hold at all times, but you may substitute $t = 0$ if that makes things simpler for you. Write down a relationship that must hold between E_{i0}, E_{r0}, and E_{t0}.

Problem 129. Apply the boundary condition for the tangential component of $\mathbf{H}$ to this situation, assuming that there is no free current density at the boundary. Since the boundary is located at $z = 0$, you may substitute $z = 0$ into the waves when you write out the boundary condition. The boundary condition must hold at all times, but you may substitute $t = 0$ if that makes things simpler for you. Write down a second relationship that must hold between E_{i0}, E_{r0}, and E_{t0}.

Problem 130. Use the results of the previous two problems to solve for E_{r0} in terms of E_{i0} and to solve for E_{t0} in terms of E_{i0}. It may help to define

$$\beta = \frac{\mu_1 v_1}{\mu_2 v_2} = \frac{\mu_1 n_2}{\mu_2 n_1} = \frac{\mu_1}{\mu_2}\sqrt{\frac{\epsilon_2 \mu_2}{\epsilon_1 \mu_1}} = \sqrt{\frac{\epsilon_2 \mu_1}{\epsilon_1 \mu_2}}.$$

Express E_{r0} in terms of E_{i0} and β. Express E_{t0} in terms of E_{i0} and β.

The energy carried across a surface S per unit time is

$$P = \int_S \mathbf{S} \cdot \hat{\mathbf{n}} dA,$$

where $\hat{\mathbf{n}}$ is a unit vector perpendicular to the surface, and $\mathbf{S} = \mathbf{E} \times \mathbf{H}$ is the Poynting vector in matter. We use the symbol P because energy per unit time is power.

We wish to define a light intensity with units of power per unit area. For this purpose it is useful, as in the integral above, to define the intensity with respect to a surface with unit normal vector $\hat{\mathbf{n}}$. We define the *intensity* I at a surface to be the magnitude of the time average of $\mathbf{S} \cdot \hat{\mathbf{n}}$ at a spot on the surface.

Suppose f is a periodic function of time t with period T. The time-averaged value of f is

$$\langle f \rangle = \frac{1}{T} \int_0^T f(t)\, dt.$$

We use angle brackets to indicate the time average.

We define the intensity to be

$$I = \left| \langle \mathbf{S} \cdot \hat{\mathbf{n}} \rangle \right|.$$

Problem 131. Find the time average of the function $f(t) = A \cos \omega t$, where A and ω are constants. Does the answer make sense?

Problem 132. Find the time average of the function $f(t) = A \cos^2 \omega t$, where A and ω are constants. Does the answer make sense?

Problem 133. Find the intensities of the incident, reflected, and transmitted waves. What relationship do they satisfy? What is the physical interpretation of this relationship?

The reflection coefficient R is the ratio of reflected light intensity to incident light intensity. The transmission coefficient T is the ratio of transmitted light intensity to incident light intensity.

Problem 134. Write expressions for the reflection and transmission coefficients in terms of β. What relationship do they satisfy?

Problem 135. When we wrote the expression

$$\mathbf{E}_i(\mathbf{r}, t) = E_{i0} \cos(k_1 z - \omega t)\hat{\mathbf{i}}$$

for the incident wave, we were actually choosing one of the four modes of the EM field that propagate in direction $\mathbf{k} = k_1\hat{\mathbf{k}}$. (Remember, in our notation $\hat{\mathbf{k}}$ is a unit vector in the z direction, and not necessarily in the direction of $\mathbf{k}$.) Write down incident waves for the electric and magnetic fields for the other three modes. What would happen if we used one of these other modes as our incident wave?

3.2 Oblique Incidence

Next, we investigate what happens when the incident wave hits the interface plane at an angle. Let us choose the following expressions for our waves.

$$\mathbf{E}_i(\mathbf{r}, t) = \mathbf{E}_{i0} \cos(\mathbf{k}_i \cdot \mathbf{r} - \omega_i t) \tag{3.1}$$
$$\mathbf{E}_r(\mathbf{r}, t) = \mathbf{E}_{r0} \cos(\mathbf{k}_r \cdot \mathbf{r} - \omega_r t) \tag{3.2}$$
$$\mathbf{E}_t(\mathbf{r}, t) = \mathbf{E}_{t0} \cos(\mathbf{k}_t \cdot \mathbf{r} - \omega_t t) \tag{3.3}$$

Problem 136. Write down the corresponding magnetic fields for the incident, reflected, and transmitted waves.

Problem 137. Given that there is no free charge or current at the surface, write the boundary conditions for each physical quantity that remains continuous over the boundary. Do not set $t = 0$ as we did in the case of normal incidence. (There should be four boundary conditions that lead to equations.)

3.2.1 A Frequency Lemma

We now present a mathematical result that will be useful for some of the upcoming problems. It says roughly that if you add two cosine functions together and get another cosine, then the frequencies of all three cosine functions must be the same.

Frequency Lemma. *If $A \neq 0$, $B \neq 0$, $C \neq 0$, $a > 0$, $b > 0$, $c > 0$, and*

$$A \cos ax + B \cos bx = C \cos cx \tag{3.4}$$

for all x, then $A + B = C$ and $a = b = c$.

Proof. Set $x = 0$ in (3.4) to obtain

$$A + B = C. \tag{3.5}$$

Taking two derivatives of (3.4) gives

$$-a^2 A \cos ax - b^2 B \cos bx = -c^2 C \cos cx. \tag{3.6}$$

Negating and setting $x = 0$ in (3.6) gives

$$a^2 A + b^2 B = c^2 C. \tag{3.7}$$

Taking two derivatives of (3.6) gives

$$a^4 A \cos ax + b^4 B \cos bx = c^4 C \cos cx. \tag{3.8}$$

Setting $x = 0$ in (3.8) gives

$$a^4 A + b^4 B = c^4 C. \tag{3.9}$$

Multiplying (3.5) by (3.9) gives

$$a^4 A^2 + a^4 AB + b^4 AB + b^4 B^2 = c^4 C^2. \tag{3.10}$$

Squaring equation (3.7) gives

$$a^4 A^2 + 2a^2 b^2 AB + b^4 B^2 = c^4 C^2. \tag{3.11}$$

Subtracting (3.11) from (3.10) gives

$$a^4 AB - 2a^2 b^2 AB + b^4 AB = 0 \tag{3.12}$$

or

$$(a^2 - b^2)^2 AB = 0. \tag{3.13}$$

Since $A \neq 0$ and $B \neq 0$, we have $a^2 = b^2$. Since $a > 0$ and $b > 0$, we have $a = b$.

Plug $b = a$ into (3.7) to obtain

$$a^2 C = c^2 C.$$

Since $C \neq 0$, we have $a^2 = c^2$. Since $a > 0$ and $c > 0$, we have $a = c$. This completes the proof that $a = b = c$. □

Problem 138. Using the boundary conditions and the frequency lemma, show that $\omega_i = \omega_r = \omega_t$ in equations (3.1)–(3.3). Since the frequencies are all the same, we will use ω from now on to refer to all of them.

Problem 139. What is the relationship between the wavenumbers k_i, k_r, and k_t (the wavenumber is the magnitude of the wavevector).

Problem 140. Apply the boundary condition for the tangential component of $\mathbf{E}$ to this situation.

Problem 141. Show that the x-components of $\mathbf{k}_i$, $\mathbf{k}_r$, and $\mathbf{k}_t$ must all be the same.

Problem 142. Must the y-components of $\mathbf{k}_i$, $\mathbf{k}_r$, and $\mathbf{k}_t$ all be the same? Why or why not? Must the z-components of $\mathbf{k}_i$, $\mathbf{k}_r$, and $\mathbf{k}_t$ all be the same? Why or why not?

At this point, let's reorient our x and y axes to make our lives simpler. Let's choose the direction of our x axis so that $\mathbf{k}_i$ lies in the xz plane.

Problem 143. Explain why $\mathbf{k}_r$ and $\mathbf{k}_t$ must also lie in the xz plane.

Under these conditions, the xz plane is called the *plane of incidence*. The *angle of incidence*, θ_i, is the angle between the vector $\mathbf{k}_i$ and the z-axis. The *angle of reflection*, θ_r, is the angle between the vector $\mathbf{k}_r$ and the z-axis. The *angle of refraction*, θ_t, is the angle between the vector $\mathbf{k}_t$ and the z-axis. All three of these angles are between 0 and 90 degrees.

Problem 144. Make a picture of the plane of incidence, showing the vectors $\mathbf{k}_i$, $\mathbf{k}_r$, $\mathbf{k}_t$, and the angles θ_i, θ_r, and θ_t.

Problem 145. Show that the angle of incidence equals the angle of reflection.

$$\theta_i = \theta_r$$

(Hint: consider what you have shown about the relationship between the components of the wavevectors and the magnitudes of the wavevectors.)

Problem 146. Prove Snell's law.

$$n_1 \sin\theta_i = n_2 \sin\theta_t$$

(Hint: consider what you have shown about the relationship between the components of the wavevectors and the magnitudes of the wavevectors.)

Our next job is to relate the amplitudes of the reflected and transmitted waves to the amplitude of the incident wave, just as we did in the case of normal incidence.

Let us choose the following expressions for our waves.

$$\begin{aligned}
\mathbf{E}_i(\mathbf{r},t) &= E_{i0}\cos(\mathbf{k}_i\cdot\mathbf{r}-\omega t)\hat{\mathbf{j}}\\
\mathbf{E}_r(\mathbf{r},t) &= E_{r0}\cos(\mathbf{k}_r\cdot\mathbf{r}-\omega t)\hat{\mathbf{j}}\\
\mathbf{E}_t(\mathbf{r},t) &= E_{t0}\cos(\mathbf{k}_t\cdot\mathbf{r}-\omega t)\hat{\mathbf{j}}
\end{aligned}$$

Problem 147. Write down the corresponding magnetic fields for the incident, reflected, and transmitted waves.

Problem 148. Make a picture of the plane of incidence, including the boundary between the two materials; the three wavevectors; the three angles θ_i, θ_r, and θ_t; and the directions of the electric and magnetic fields for each of the three waves.

Problem 149. Apply the boundary condition for the tangential component of **E** to this situation. Write down a relationship that must hold between E_{i0}, E_{r0}, and E_{t0}.

Problem 150. Apply the boundary condition for the tangential component of **H** to this situation. Write down a second relationship that must hold between E_{i0}, E_{r0}, and E_{t0}. Express your results in terms of β defined earlier and

$$\alpha = \frac{\cos\theta_t}{\cos\theta_i}.$$

What is this α and this β that we have defined? They are just a shorthand notation to save us some writing, but we should remember two things. First, if we know the material parameters ϵ_1, μ_1, ϵ_2, and μ_2, then we know β. So β is just an important combination of material parameters. Second, if we know the material parameters and the angle of incidence, then we know α.

Problem 151. Write an expression for α in terms of the material parameters and the angle of incidence.

Problem 152. Apply the boundary condition for the normal component of $\mathbf{B}$ to this situation. Write down a third relationship that must hold between E_{i0}, E_{r0}, and E_{t0}.

Problem 153. Are these three relationships independent? Explain.

Problem 154. Use the results of the previous problems to solve for E_{r0} in terms of E_{i0} and to solve for E_{t0} in terms of E_{i0}.

Problem 155. Find the intensities of the incident, reflected, and transmitted waves. What relationship do they satisfy? What is the physical interpretation of this relationship?

Problem 156. Write expressions for the reflection and transmission coefficients in terms of α and β. What relationship do they satisfy?

Problem 157. Consider an air-glass interface with $\mu_1 = \mu_2 = \mu_0$, $n_1 = 1$, and $n_2 = 1.5$. Make a plot of the reflection coefficient as a function of incidence angle.

Problem 158. Consider a glass-air interface with $\mu_1 = \mu_2 = \mu_0$, $n_1 = 1.5$, and $n_2 = 1$. Make a plot of the reflection coefficient as a function of incidence angle.

Problem 159. When we wrote the expression

$$\mathbf{E}_i(\mathbf{r}, t) = E_{i0} \cos(\mathbf{k}_i \cdot \mathbf{r} - \omega t)\hat{\mathbf{j}}$$

for the incident wave, we were actually choosing one of the four modes of the EM field that propagate in direction $\mathbf{k}_i$. Write down incident waves for the electric and magnetic fields for the other three modes. What would happen if we used one of these other modes as our incident wave?

The previous analysis, in which the electric field is polarized perpendicular to the plane of incidence, is called TE (transverse electric) polarization. The polarization in which the magnetic field points perpendicular to the plane of incidence is called TM (transverse magnetic) polarization.

Problem 160. Write down the electric and magnetic fields for the incident, reflected, and transmitted waves for the case of TM polarization.

Problem 161. Make a picture of the plane of incidence, including the boundary between the two materials; the three wavevectors; the three angles θ_i, θ_r, and θ_t; and the directions of the electric and magnetic fields for each of the three waves.

Problem 162. Apply the boundary condition for the tangential component of $\mathbf{E}$ to this situation. Write down a relationship that must hold between E_{i0}, E_{r0}, and E_{t0}.

Problem 163. Apply the boundary condition for the tangential component of $\mathbf{H}$ to this situation. Write down a second relationship that must hold between E_{i0}, E_{r0}, and E_{t0}.

Problem 164. Apply the boundary condition for the normal component of $\mathbf{D}$ to this situation. Write down a third relationship that must hold between E_{i0}, E_{r0}, and E_{t0}.

Problem 165. Are these three relationships independent? Explain.

Problem 166. Use the results of the previous problems to solve for E_{r0} in terms of E_{i0} and to solve for E_{t0} in terms of E_{i0}. It may help to express your results in terms of α and β.

Problem 167. Find the intensities of the incident, reflected, and transmitted waves. What relationship do they satisfy? What is the physical interpretation of this relationship?

Problem 168. Write expressions for the reflection and transmission coefficients in terms of α and β. What relationship do they satisfy?

Problem 169. Consider an air-glass interface with $\mu_1 = \mu_2 = \mu_0$, $n_1 = 1$, and $n_2 = 1.5$. Make a plot of the reflection coefficient for a TM polarized wave as a function of incidence angle.

Problem 170. Consider a glass-air interface with $\mu_1 = \mu_2 = \mu_0$, $n_1 = 1.5$, and $n_2 = 1$. Make a plot of the reflection coefficient for a TM polarized wave as a function of incidence angle.

Problem 171. We have material 1 with index of refraction 1 filling the entire region of space $z < 0$. We have material 2 with index of refraction 4/3 filling the region $z > 0$. The plane $z = 0$ forms the boundary between the two media, and has no free surface charge density. Both materials have permeability μ_0. A plane EM wave in material 1 has electric field

$$\mathbf{E}_i = (100\ \text{V/m}) \cos\left[\frac{(12x + 9z)}{1500\ \text{nm}} - (3 \times 10^{15}\ \text{rad/s})t\right] \hat{\mathbf{y}}.$$

(a) What is the wavelength of this wave?

(b) Write a magnetic field for this wave.

(c) As this EM wave hits the boundary, there will be a reflected wave. Write the electric and magnetic fields for the reflected wave.

(d) As this EM wave hits the boundary, there will be a transmitted wave. Write the electric and magnetic fields for the transmitted wave.

(e) Confirm that the tangential component of **E** is continuous across the boundary.

(f) Confirm that the tangential component of **H** is continuous across the boundary.

(g) Confirm that the normal component of **B** is continuous across the boundary.

(h) Confirm that the incident intensity is equal to the reflected intensity plus the transmitted intensity.

Problem 172. For one of the polarizations (TE or TM), there is an angle (called Brewster's angle) at which there is no reflected light. Which polarization has this feature? Find an expression for Brewster's angle in terms of the material parameters n_1 and n_2 in the simplified case in which $\mu_1 = \mu_2$.

Problem 173. Get some polarizing film from the lab and show me Brewster's angle.

Problem 174. Another interesting angle is the critical angle for total internal reflection. Under what conditions does this occur? Find an expression for this angle. What goes to zero when the angle of incidence is greater than this angle? Show how this idea is contained in the expressions we have derived. Do both polarizations have the same critical angle?

Chapter 4

Special Relativity

We will endeavor to derive the results of special relativity from the following two postulates.

1. The laws of physics are the same in all inertial reference frames.
2. The speed of light is the same in all inertial reference frames.

As you know from your previous study of special relativity in Physics 211, we must give up the universality of time. Observers in different inertial reference frames will not agree on the time that elapses between two events. This raises the natural question of what else they might not agree about. (You may recall that they don't agree about lengths either.) I find that to make progress in our thinking, we must assume two more things in addition to the big principles above.

3. If frame A measures frame B to be moving with velocity $\mathbf{v}$, then frame B measures frame A to be moving with velocity $-\mathbf{v}$.
4. Observers in frames A and B agree on distances that they measure in directions *perpendicular* to the relative velocity $\mathbf{v}$ of the frames.

4.1 Lorentz Transformations

4.1.1 Time dilation

Consider the following situation. Train car A is stationary in an inertial reference frame. This does not mean that train car A is stationary with respect to the train tracks, but just that train car A serves as a good inertial reference frame. We set up a coordinate system (x_A, y_A, z_A) on train car A, with the origin at the center of the floor of the train car. The x_A direction points along the train tracks (say east), the y_A direction points north, and the z_A direction points up. There is also a clock in train car A, and the time on that clock is t_A.

Train car B moves at a constant velocity v with respect to train car A. Train car B's position in frame A is given by

$$\begin{aligned} x_A^{\text{car B}}(t_A) &= vt_A \\ y_A^{\text{car B}}(t_A) &= 0 \\ z_A^{\text{car B}}(t_A) &= 0. \end{aligned}$$

The superscript "car B" means that these are the x, y, and z coordinates of car B. The subscript A means that we are using the coordinate system and the clock of frame A to make our measurements and to describe the location of car B.

Notice that train car B is sitting in the same place as train car A at $t_A = 0$. This is a little unphysical. If you want, you can think of train car B as an imaginary car.

We set up a coordinate system (x_B, y_B, z_B) on train car B, with the origin at the center of the floor of the train car. The x_B direction points along the train tracks (say east), the y_B direction points north, and the z_B direction points up. There is also a clock in train car B, and the time on that clock is t_B.

So, we have two different frames of reference (or two different spacetime coordinate systems) from which to view and describe things that happen.

One last detail: We already know that $t_A = 0$ when the two cars are right on top of each other. Let's set the clock on train car B so that $t_B = 0$ also when the two cars are right on top of each other.

Consider the following two events.

Event 1: There is a light bulb at the center of the floor (the origin) of train car A. The bulb turns on when the two cars are right on top of each other.

Event 2: The light hits the center of the ceiling of train car A, a distance h above the floor of car A. There is a small light detector in the center of the ceiling of car A.

Problem 175. Write down spacetime coordinates for Event 1 and Event 2 in frame A.

Problem 176. Write down spacetime coordinates for Event 1 and Event 2 in frame B.

Problem 177. Look at the time that elapses between events 1 and 2 (a) in frame A, and (b) in frame B. Are the time intervals the same? If not which is longer?

We see that observers in different inertial frames see things differently. What we would like to have is way to figure out how things would look in frame B if we know how they look in frame A. More precisely, we seek a way to find the spacetime coordinates (x,y,z,t) of an event in frame B if we know the spacetime coordinates of the event in frame A. This is what the Lorentz transformation does.

In the following set of problems, we suppose that the Lorentz transformation should be linear, and derive what it must look like to accomodate the results we've already found.

Problem 178. We postulate that the Lorentz transformation is a linear transformation between coordinates.

$$\begin{aligned} ct_B &= \lambda_{00} ct_A + \lambda_{01} x_A \\ x_B &= \lambda_{10} ct_A + \lambda_{11} x_A \end{aligned}$$

Use the events above to find λ_{00} and λ_{10}.

4.1.2 A second experiment

Now we will do a second experiment with our two train cars.

Consider the following events.

Event 1: There is a light bulb at the center of the floor (the origin) of train car A. The bulb turns on when the two cars are right on top of each other.

Event 2: The light hits a mirror mounted on one end of train car A, at $x_A = L$. Let's focus our attention on the light that travels horizontally along the floor of the train car.

Event 3: After reflecting from the mirror in train car A, the light hits a detector at the origin in train car B.

Problem 179. Write down spacetime coordinates for Events 1, 2, and 3 in frame A.

Problem 180. Write down spacetime coordinates for Events 1, 2, and 3 in frame B. Let us say that the location of the mirror when the light hits it is $x_B = l$.

Problem 181. Earlier, we found expressions for λ_{00} and λ_{10} in the Lorentz transformation.

$$\begin{aligned} ct_B &= \lambda_{00} ct_A + \lambda_{01} x_A \\ x_B &= \lambda_{10} ct_A + \lambda_{11} x_A \end{aligned}$$

Using the events in this most recent experiment, find expressions for l, λ_{01} and λ_{11} in terms of v and L.

Problem 182. Write the Lorentz transformation for frame B moving with velocity $\mathbf{v} = v\hat{\mathbf{i}}$ with respect to frame A. Write it in matrix form.

$$\begin{bmatrix} ct_B \\ x_B \\ y_B \\ z_B \end{bmatrix} = \begin{bmatrix} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{bmatrix} \begin{bmatrix} ct_A \\ x_A \\ y_A \\ z_A \end{bmatrix}$$

You may wish to use the shorthand expressions $\beta = v/c$ and $\gamma = 1/\sqrt{1 - v^2/c^2}$.

Problem 183. Algebraically manipulate the Lorentz transformation to give equations for ct_A and x_A in terms of ct_B and x_B. Then write it in matrix form.

$$\begin{bmatrix} ct_A \\ x_A \\ y_A \\ z_A \end{bmatrix} = \begin{bmatrix} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{bmatrix} \begin{bmatrix} ct_B \\ x_B \\ y_B \\ z_B \end{bmatrix}$$

You may wish to use the shorthand expressions $\beta = v/c$ and $\gamma = 1/\sqrt{1 - v^2/c^2}$. How could we have guessed this result without doing the algebra?

4.1.3 Length contraction

Consider two events in spacetime. In frame A, the distance between them is Δx_A (let's not worry about the y and z directions), and in frame B, the distance between the two events is Δx_B. The time between the two events is Δt_A in frame A and Δt_B in frame B. With the Lorentz transformation, we know exactly how these intervals are related.

$$
\begin{aligned}
c\Delta t_B &= \gamma c\Delta t_A - \beta\gamma\Delta x_A \\
\Delta x_B &= -\beta\gamma c\Delta t_A + \gamma\Delta x_A
\end{aligned}
$$

Consider two events that occur at the same time in frame A. In that case, we have $\Delta t_A = 0$, and so we have

$$\Delta x_B = \gamma\Delta x_A$$

Problem 184. This isn't length contraction, it's length dilation. What's wrong with this?

The trouble here is that *length* is not the spatial distance between two events in spacetime. For a person in a reference frame to regard the spatial distance between two events to be a length, those events must occur at the same time *in that person's frame*. To make this point clear, suppose that I have information (expressed in position and time coordinates in my reference frame) about the position of the front of a car at a certain time, and the position of the rear of the car 5 seconds later. From that information, can I determine the length of the car? No, I can't, unless I happen to know that the car is stationary during that period.

But if I know the position of the front of the car at a certain time, and the position of the rear of the car at the same time (in my reference frame), then I can compute the length of the car (by subtracting), even if the car is moving.

So, it is not appropriate to speak about the length between events in spacetime (important as events in spacetime are). It is appropriate to speak about length as being the distance between two particles, or two small objects, at a single point in time. Particles are not events in spacetime. Particles are described by curves in spacetime that give their position at each point in time. These curves in spacetime are sometimes called *world lines*. We normally think of length as being a property of an extended object that takes up some space. Now we are ready to say what length is.

The *length* of an object from the perspective of a reference frame is the distance between the positions of two particles at opposite ends of the object, taken at the same time in that reference frame.

Problem 185. Return again to the two train cars A and B that we started with at the beginning of the chapter. Using coordinates in frame B, write an expression for the position of the east end of train car B as a function of time. Using the Lorentz transformation and this equation, obtain an equation for the position of the east end of train car B as a function of time *using the position and time coordinates of frame A*.

Problem 186. A spacetime diagram is a graph of ct vs. x in a particular reference frame. It is conventional when doing relativity to put ct on the vertical axis and x on the horizontal axis. Make a spacetime diagram in frame A that shows the following:

- a line representing the east end of train car A,
- a line representing the west end of train car A,
- a line representing the east end of train car B,
- a line representing the west end of train car B.

For this diagram, take the speed v of train car B with respect to train car A to be $v = \frac{4}{5}c$.

Problem 187. Make a spacetime diagram in frame B that shows the following:

- a line representing the east end of train car A,
- a line representing the west end of train car A,
- a line representing the east end of train car B,
- a line representing the west end of train car B.

For this diagram, take the speed v of train car B with respect to train car A to be $v = \frac{4}{5}c$.

Problem 188. Show how to use the Lorentz transformation to find the length of train car B in the frame of train car A.

Conclusion: The length of a moving object shrinks by a factor of γ compared with its length in its own frame. (We are assuming here an object moving with constant velocity.)

4.1.4 The invariant interval

Suppose there are two events in spacetime. In frame A, the differences in coordinates between the two events are $c\Delta t_A$, Δx_A, Δy_A, and Δz_A.

There is an interesting quantity, called the invariant interval, or the spacetime interval, I.

$$I = -(c\Delta t_A)^2 + (\Delta x_A)^2 + (\Delta y_A)^2 + (\Delta z_A)^2$$

The interesting thing about this particular combination of quantities is that every frame agrees about this number for these two events. That is what we mean when we use the term *invariant*. All frames agree about the value of the invariant interval. So, if frame B has coordinates for these same two spacetime events, and the differences between coordinate values in frame B are $c\Delta t_B$, Δx_B, Δy_B, and Δz_B, then

$$\begin{aligned} I &= -(c\Delta t_A)^2 + (\Delta x_A)^2 + (\Delta y_A)^2 + (\Delta z_A)^2 \\ &= -(c\Delta t_B)^2 + (\Delta x_B)^2 + (\Delta y_B)^2 + (\Delta z_B)^2 \end{aligned}$$

even though, in general, $c\Delta t_A \neq c\Delta t_B$, $\Delta x_A \neq \Delta x_B$, etc.

Problem 189. Using the Lorentz transformation that we developed, show that the invariant interval is the same in frames A and B.

You may notice that the invariant interval is similar, except for the time term, to the square of the distance between points in three-dimensional Euclidean space. It is a worthwhile analogy to pursue. In 3D space, we can change coordinates, changing our perspective, by using a second set of coordinates rotated with respect to the first. Those two coordinate systems disagree about the coordinates, and the differences between coordinates for two points in 3D space. But, those two coordinate systems agree on the distance between points.

In spacetime, Lorentz transformations play the role of rotations, and the invariant interval plays the role of distance. This analogy is summarized in the table below.

	3D Euclidean space	Spacetime
Transformation	Rotation	Lorentz transformation
Invariant	Distance between points	Invariant interval

In 3D space, the square of the distance between two points is always positive, so we can always take the square root to get a thing we call the distance. In spacetime, the invariant interval between two spacetime events can be negative, zero, or positive, so we can't just take the sqaure root to get a spacetime "distance."

There are three different ways that two spacetime points can be in relation to each other. These relationships have physical significance, so we give them names. The table below shows the three cases.

$I < 0$	time-like
$I = 0$	light-like
$I > 0$	space-like

If two events are time-like related, or time-like separated, then (roughly speaking) these two events are farther apart in time than they are in space. There is a frame in which the two events occur at the same place. Observers from different frames do not agree on the time that elapses between the two events, but all observers agree about which event happened first. Two events experienced by a particle with mass must be time-like related. In this case, we define the *proper time* between the events to be the time that elapses in the reference frame where both events occur at the same place. For time-like related events, proper time plays the role that distance plays in 3D space.

If two events are light-like related, or light-like separated, then these two events lie on a path that light could take. Unlike in 3D space, where the distance between points is zero only for two points that are really the same point, the invariant interval between different points in spacetime is zero whenever they are light-like related. (Thus, in the language of mathematicians, spacetime fails to be a "metric space.") For light-like related events, there is no good analogy for the role that distance plays in 3D space.

If two events are space-like related, or space-like separated, then (roughly speaking) these two events are farther apart in space than they are in time. There is a frame in which the two events occur at the same time. There are also frames in which event 1 occurs before event 2, and other frames in which event 1 occurs after event 2. Space-like events are not causally related, meaning that one event can not have an influence on the other event (because not even light can get from one event to the other). It also seems reasonable that the two events not be causally related, since observers do not agree about which event occurred first, and the basic idea behind causality is that a cause must occur earlier in time than an effect. In the case of space-like related events, we define the *proper length* between the events to be the spatial distance between the events in the reference frame where both events occur at the same time. For space-like related events, proper length plays the role that distance plays in 3D space.

Problem 190. For each of the following pairs of events, say whether the events are time-like, light-like, or space-like related. If the events are time-like related, find the velocity at which a reference frame must travel (with respect to the given reference frame), so that both events occur at the same place. If the events are space-like related, find the velocity at which a reference frame must travel (with respect to the given reference frame), so that both events occur at the same time.

(a) Event 1: $ct = 10$ m, $x = 10$ m
Event 2: $ct = 5$ m, $x = 6$ m

(b) Event 1: $ct = 4$ m, $x = 6$ m
Event 2: $ct = 6$ m, $x = -4$ m

(c) Event 1: $ct = 2$ m, $x = 8$ m
Event 2: $ct = -8$ m, $x = 18$ m

(d) Event 1: $ct = 10$ m, $x = 9$ m, $y = 2$ m
Event 2: $ct = 5$ m, $x = 6$ m, $y = 6$ m

(e) Event 1: $ct = 5$ m, $x = 5$ m, $y = 5$ m
Event 2: $ct = 3$ m, $x = 3$ m, $y = 3$ m

(f) Event 1: $ct = 8$ m, $x = 9$ m, $y = 2$ m
Event 2: $ct = 5$ m, $x = 7$ m, $y = 4$ m

(g) Event 1: $ct = 8$ m, $x = 8$ m, $y = 4$ m, $z = 4$ m
Event 2: $ct = 5$ m, $x = 6$ m, $y = 6$ m, $z = 5$ m

4.2 Relativity in Electrodynamics

How do electric and magnetic phenomena look from different reference frames? That is the main question of this section. The answer will turn out to be that the physical quantities we are interested in fall into three categories (to be called scalars, 4-vectors, and second rank tensors), based on how they transform under a Lorentz transformation.

4.2.1 The invariance of electric charge

There is very good evidence that electric charge is a Lorentz invariant. That means that all reference frames agree on the amount of electric charge in a given situation. Unlike distance, time, and even mass, charge does not get multiplied or divided by a factor like γ when we change frames. There is no charge contraction or charge dilation. Charge stays the same. And that's kind of comforting. At least *something* stays the same.

The evidence for this comes from precise experiments about the electrical neutrality of helium and the electrical neutrality of the diatomic hydrogen molecule. Each of these particles has two protons and two electrons. The protons in helium, being tightly bound in the nucleus, have a large kinetic energy and a large RMS speed. If the motion of charged particles affected the electrical charge of the particle, one would not expect both diatomic hydrogen and helium to be electrically neutral to such a high degree. (Experiments have been performed that show the electrical neutrality of each of them to about 1 part in 10^{20}. See Purcell for more info.)

4.2.2 An infinite plate of charge

Suppose that in frame B there is a stationary plate of charge with uniform surface charge density σ in the xy plane.

Problem 191. What is the electric field produced by this plate of charge?

From the perspective of frame A, the plate of charge is moving, so there is a surface current density in addition to a surface charge density.

Problem 192. What is the surface charge density produced by the plate, from frame A's perspective?

Problem 193. What is the surface current density produced by the plate, from frame A's perspective?

Problem 194. What is the electric field of the plate, as seen by frame A?

Problem 195. What is the magnetic field of the plate, as seen by frame A?

4.3 4-vectors

In Newtonian mechanics, we use vectors with 3 components (x, y, z) because space has three dimensions. In special relativity, we use vectors with 4 components because spacetime has 4 dimensions.

We will use tensor notation to denote 4-vectors and to denote the second-rank tensors that we will introduce later. In tensor notation, the components of a vector are labeled with superscripts. In addition to vectors, there are also beasts called dual vectors, whose components are labeled with subscripts. It is an irritation that superscripts are used, because it conflicts with our usual notation that a superscript 2 represents squaring and 3 represents cubing. Unfortunately, the use of tensor notation is fairly standard among people that do relativity. It's actually not a bad notation once you get used to it.

There is nothing inherently relativistic about tensor notation. Tensor notation could be used, and sometimes is used, in Newtonian mechanics. In that case, the superscripts and subscripts go from 1 to 3. In relativity, they go from 0 to 3.

4.3.1 The position 4-vector

We package position (a Newtonian 3-vector) and time (a Newtonian scalar) together into a position 4-vector.

$$\begin{aligned} x^0 &= ct & \qquad x_0 &= -ct \\ x^1 &= x & x_1 &= x \\ x^2 &= y & x_2 &= y \\ x^3 &= z & x_3 &= z \end{aligned}$$

We can put all of this together into a box.

$$x^\mu = \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix} = \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix}$$

4.3.2 The proper time

The proper time is not a 4-vector, but now is a convenient time to talk about it. The proper time experienced by a particle or an object is the time that elapses on a clock or watch carried by that object. We use the Greek letter τ for proper time.

In special relativity, each frame of reference has its own notion of how time elapses. We use symbols like t and t' (or x^0 and x'^0) to describe the time measured by observers in a particular reference frame. The proper time is not necessarily the same as the time measured by any observer in an inertial reference frame. The reason for this is that particles can accelerate, while inertial reference frames cannot. The proper time is important precisely because it is not associated with a particular reference frame.

4.3.3 The 4-velocity

In Newtonian mechanics, the 3-velocity of a particle is defined to be the derivative of the position vector of the particle with respect to time. In special relativity, we also want to take the derivative of a position vector with respect to a time to give us a velocity. We want to have a 4-vector for our velocity, so it seems natural to take the derivative of the position 4-vector. But in special relativity every hot dog salesman has his own opinion of what the time should be. We need to take a derivative with respect to time. Which time should we use? The best way to proceed is to use the proper time.

In special relativity, the 4-velocity of a particle is the derivative of the position 4-vector of the particle with respect to the *proper time* of the particle.

$$v^\alpha = \frac{dx^\alpha}{d\tau}$$

$$\begin{bmatrix} v^0 \\ v^1 \\ v^2 \\ v^3 \end{bmatrix} = \begin{bmatrix} \frac{dx^0}{d\tau} \\ \frac{dx^1}{d\tau} \\ \frac{dx^2}{d\tau} \\ \frac{dx^3}{d\tau} \end{bmatrix} = \begin{bmatrix} c\frac{dt}{d\tau} \\ \frac{dx}{d\tau} \\ \frac{dy}{d\tau} \\ \frac{dz}{d\tau} \end{bmatrix}$$

How does this relate to the 3-velocity which we know and love?

Problem 196. A particle has coordinates

$$x^\mu = \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix} = \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix},$$

and 3-velocity $\mathbf{v} = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} + v_z\hat{\mathbf{k}}$ with respect to a particular frame of reference. Consider two nearby events along the particle's world line. If the time between these two events in the frame is dt, and the elapsed proper time for the particle is $d\tau$, what is the relationship between $d\tau$ and dt? (Hint: Imagine that the particle is momentarily at rest in frame B, traveling at velocity $\mathbf{v}$ with respect to frame A.)

Problem 197. Write expressions for the four components of the 4-velocity in terms of the components v_x, v_y, and v_z of the 3-velocity.

4.3.4 More 4-vectors

There are three other 4-vectors that are used to describe physical quantities associated with a particle.

The 4-momentum of a particle is equal to the product of the particle's rest mass and its 4-velocity.

$$p^\mu = mv^\mu$$

Problem 198. What is the meaning of the time component of the 4-momentum?

The 4-force is the 4-vector that plays the role of force in relativity. We will use the notation K^μ for 4-force. The relativistic version of Newton's second law is

$$K^\mu = \frac{dp^\mu}{d\tau},$$

where K^μ is the *net* 4-force in this equation. The spatial components of the 4-force are related to the components of the 3-force by

$$\begin{aligned} K^1 &= \gamma F_x \\ K^2 &= \gamma F_y \\ K^3 &= \gamma F_z, \end{aligned}$$

where

$$\gamma = \frac{1}{\sqrt{1 - \frac{v_x^2}{c^2} - \frac{v_y^2}{c^2} - \frac{v_z^2}{c^2}}},$$

is based on the velocity of the particle and not the velocity between two frames.

Problem 199. Show, for a particle with rest mass m, 3-velocity $\mathbf{u}$, and 3-acceleration $\mathbf{a}$, that the relativistic version of Newton's second law is

$$\mathbf{F} = \frac{m}{\sqrt{1 - u^2/c^2}} \left[\mathbf{a} + \frac{\mathbf{u}(\mathbf{u} \cdot \mathbf{a})}{c^2 - u^2} \right],$$

when expressed in 3-vector notation. (Here $\mathbf{F}$ is the net 3-force on the particle.)

The last 4-vector associated with a particle is the 4-acceleration a^μ. The 4-acceleration is related to the 4-velocity by

$$a^\mu = \frac{dv^\mu}{d\tau}.$$

We could also write Newton's second law as

$$K^\mu = ma^\mu,$$

where m is the rest mass of the particle.

4.3.5 Types of physical quantities

Here is a summary of physical quantities from the perspective of relativity.

- Scalars (Rank-0 4-Tensors)
  - Rest mass of a particle
  - Charge of a particle
- 4-Vectors (Rank-1 4-Tensors)
  - 4-velocity of a particle
  - 4-momentum of a particle

  - 4-force on a particle
  - 4-acceleration of a particle
  - 4-position of a particle (in dispute)

- Scalar Fields (Rank-0 4-Tensor Fields)

  - I can't think of any. Can you?

- 4-Vector Fields (Rank-1 4-Tensor Fields)

  - Current density
  - Vector potential

- Rank-2 4-Tensor Fields

  - EM Field Tensor
  - Metric Tensor
  - Stress-Energy-Momentum Tensor
  - Lorentz Transformation (in dispute)

- Rank-4 4-Tensor Fields

  - Curvature Tensor for Spacetime (in General Relativity)

4.3.6 The Current Density 4-vector

We package current density (a Newtonian 3-vector) and charge density (a Newtonian scalar) together into a current density 4-vector.

$$J^\mu = \begin{bmatrix} J^0 \\ J^1 \\ J^2 \\ J^3 \end{bmatrix} = \begin{bmatrix} c\rho \\ J_x \\ J_y \\ J_z \end{bmatrix}$$

Another piece of tensor notation that people use is to write ∂_μ to mean the partial derivative with respect to the μth variable. So, we have

$$\begin{aligned}
\partial_0 &= \frac{\partial}{\partial x^0} = \frac{1}{c}\frac{\partial}{\partial t} \\
\partial_1 &= \frac{\partial}{\partial x^1} = \frac{\partial}{\partial x} \\
\partial_2 &= \frac{\partial}{\partial x^2} = \frac{\partial}{\partial y} \\
\partial_3 &= \frac{\partial}{\partial x^3} = \frac{\partial}{\partial z}.
\end{aligned}$$

4.3.7 How 4-vectors transform

How do 4-vectors transform under a Lorentz transformation? Suppose we have a 4-vector

$$a^\mu = \begin{bmatrix} a^0 \\ a^1 \\ a^2 \\ a^3 \end{bmatrix},$$

as viewed from frame A. How does it look in frame B? We need to apply a Lorentz transformation. Let us write the Lorentz transformation matrix that we found before in our new tensor notation.

$$\Lambda^\mu{}_\nu = \begin{bmatrix} \Lambda^0{}_0 & \Lambda^0{}_1 & \Lambda^0{}_2 & \Lambda^0{}_3 \\ \Lambda^1{}_0 & \Lambda^1{}_1 & \Lambda^1{}_2 & \Lambda^1{}_3 \\ \Lambda^2{}_0 & \Lambda^2{}_1 & \Lambda^2{}_2 & \Lambda^2{}_3 \\ \Lambda^3{}_0 & \Lambda^3{}_1 & \Lambda^3{}_2 & \Lambda^3{}_3 \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

When we write $\Lambda^3{}_2$, the left index 3 refers to the row number, and the right index 2 refers to the column number. Our convention for relating indices to matrices is that left means row and right means column.

Let the 4-vector

$$\bar{a}^\mu = \begin{bmatrix} \bar{a}^0 \\ \bar{a}^1 \\ \bar{a}^2 \\ \bar{a}^3 \end{bmatrix}$$

represent the 4-vector as observed from frame B. Then we know that

$$\begin{bmatrix} \bar{a}^0 \\ \bar{a}^1 \\ \bar{a}^2 \\ \bar{a}^3 \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a^0 \\ a^1 \\ a^2 \\ a^3 \end{bmatrix}.$$

We can abbreviate this equation by writing

$$\bar{a}^\mu = \sum_{\nu=0}^{3} \Lambda^\mu{}_\nu a^\nu,$$

and we will (following the *Einstein summation convention*) abbreviate this even more by writing

$$\bar{a}^\mu = \Lambda^\mu{}_\nu a^\nu.$$

The summation convention says that indices that occur twice in an expression, with one occurrence as an upper index and one occurrence as a lower index, are summed over. If the index is a greek letter, the sum is from 0 to 3.

Problem 200. Translate the equation

$$\partial_\mu J^\mu = 0$$

into traditional 3-vector notation, and give its name when you recognize it.

In summary, *all* 4-vectors transform according to the rule

$$\bar{a}^\mu = \Lambda^\mu{}_\nu a^\nu.$$

Problem 201. Suppose a ball is moving with velocity $v_b\hat{\mathbf{i}}$ in frame B. If frame B moves with velocity $v\hat{\mathbf{i}}$ in frame A, how fast does the ball move in frame A? (Hint: Use the Lorentz transformation to relate the 4-velocity of the ball in frame B to the 4-velocity of the ball in frame A.) This is called the *relativistic velocity addition rule*.

4.3.8 The Metric Tensor

Let us look at the metric tensor as our first example of a second rank tensor.

$$\eta_{\mu\nu} = \begin{bmatrix} \eta_{00} & \eta_{01} & \eta_{02} & \eta_{03} \\ \eta_{10} & \eta_{11} & \eta_{12} & \eta_{13} \\ \eta_{20} & \eta_{21} & \eta_{22} & \eta_{23} \\ \eta_{30} & \eta_{31} & \eta_{32} & \eta_{33} \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The metric tensor can be used to lower one index of a tensor. A vector is a rank-1 tensor. Suppose a^μ are the components of a 4-vector. Then

$$a_\mu = \eta_{\mu\nu} a^\nu.$$

Writing out all of the components, this looks like

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} \eta_{00} & \eta_{01} & \eta_{02} & \eta_{03} \\ \eta_{10} & \eta_{11} & \eta_{12} & \eta_{13} \\ \eta_{20} & \eta_{21} & \eta_{22} & \eta_{23} \\ \eta_{30} & \eta_{31} & \eta_{32} & \eta_{33} \end{bmatrix} \begin{bmatrix} a^0 \\ a^1 \\ a^2 \\ a^3 \end{bmatrix},$$

and then plugging in the values of $\eta_{\mu\nu}$, we have

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a^0 \\ a^1 \\ a^2 \\ a^3 \end{bmatrix} = \begin{bmatrix} -a^0 \\ a^1 \\ a^2 \\ a^3 \end{bmatrix}.$$

For any 4-vector, then, a component with an upper index is equal to the corresponding component with a lower index if the index is 1, 2, or 3 (one of the spatial indices). If the index is 0 (the time component of the 4-vector), then the component with upper index is opposite the component with lower index.

Problem 202. Evaluate $v_\mu v^\mu$ and simplify.

4.3.9 The EM Field Tensor

From the standpoint of relativity, the electric and magnetic fields combine into a single object that we call the EM field tensor. It is a second rank tensor field.

$$F^{\mu\nu} = \begin{bmatrix} F^{00} & F^{01} & F^{02} & F^{03} \\ F^{10} & F^{11} & F^{12} & F^{13} \\ F^{20} & F^{21} & F^{22} & F^{23} \\ F^{30} & F^{31} & F^{32} & F^{33} \end{bmatrix} = \begin{bmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{bmatrix}$$

Problem 203. Translate the equation

$$\partial_\nu F^{\mu\nu} = \mu_0 J^\mu$$

into traditional 3-vector notation. Do you recognize it?

Problem 204. Translate the equation

$$K^\mu = q v_\nu F^{\mu\nu}$$

into traditional 3-vector notation. Do you recognize it?

The dual EM field tensor is defined to be

$$G^{\mu\nu} = \begin{bmatrix} G^{00} & G^{01} & G^{02} & G^{03} \\ G^{10} & G^{11} & G^{12} & G^{13} \\ G^{20} & G^{21} & G^{22} & G^{23} \\ G^{30} & G^{31} & G^{32} & G^{33} \end{bmatrix} = \begin{bmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z/c & E_y/c \\ -B_y & E_z/c & 0 & -E_x/c \\ -B_z & -E_y/c & E_x/c & 0 \end{bmatrix}$$

There is a geometric relationship between the EM field tensor and the dual EM field tensor, but a statement of this relationship is beyond the scope of this course.

You can see that we get the dual tensor from the original one by making the replacements

$$\begin{aligned} \frac{\mathbf{E}}{c} &\to \mathbf{B} \\ \mathbf{B} &\to -\frac{\mathbf{E}}{c}. \end{aligned}$$

Problem 205. Translate the equation

$$\partial_\nu G^{\mu\nu} = 0$$

into traditional 3-vector notation. Do you recognize it?

4.3.10 How tensors transform

How do second rank tensors transform under a Lorentz transformation? Recall that vectors transform according to the rule

$$\bar{a}^{\mu} = \Lambda^{\mu}{}_{\nu} a^{\nu}.$$

Suppose that $t^{\mu\nu}$ is a second rank tensor. Second rank tensors transform according to the rule

$$\bar{t}^{\mu\nu} = \Lambda^{\mu}{}_{\lambda} \Lambda^{\nu}{}_{\sigma} t^{\lambda\sigma}.$$

For higher rank tensors, there is one factor of $\Lambda^{\nu}{}_{\sigma}$ for each index in the tensor (1 for rank 1, 2 for rank 2, etc.).

Problem 206. Carry out the Lorentz transformation on the EM field tensor $F^{\mu\nu}$.