Radiation

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Accelerating charges radiate.

- Accelerating charges are the source of EM radiation.
- It is not sufficient for charge to merely be moving. Stationary and moving charge has EM fields associated with it, but these EM fields are not radiation.
- Radiation is the creation of EM waves.

Maxwell equations as wave equations

If we choose the Lorenz gauge condition,

$$\mu_0\epsilon_0\frac{\partial V}{\partial t}+\vec{\nabla}\cdot\vec{\mathbf{A}}=0$$

the Maxwell equations are inhomogeneous wave equations.

$$\nabla^2 V-\frac{1}{c^2}\frac{\partial^2 V}{\partial t^2}=-\frac{1}{\epsilon_0}\rho$$

$$\nabla^2\vec{\mathbf{A}}-\frac{1}{c^2}\frac{\partial^2\vec{\mathbf{A}}}{\partial t^2}=-\mu_0\vec{\mathbf{J}}$$

Solutions to Maxwell equations

$$V(\vec{\mathbf{r}}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho\left(\vec{\mathbf{r}}', t \pm \left|\vec{\mathbf{r}} - \vec{\mathbf{r}'}\right|\right)}{\left|\vec{\mathbf{r}} - \vec{\mathbf{r}'}\right|}\, dv'$$

$$\vec{\mathbf{A}}(\vec{\mathbf{r}}, t) = \frac{\mu_0}{4\pi} \int \frac{\vec{\mathbf{J}}\left(\vec{\mathbf{r}}', t \pm \left|\vec{\mathbf{r}} - \vec{\mathbf{r}'}\right|\right)}{\left|\vec{\mathbf{r}} - \vec{\mathbf{r}'}\right|}\, dv'$$

- We are setting $c = 1$.
- The integrals are over all space.

Retarded potentials

The solutions

$$V(\vec{\mathbf{r}}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho\left(\vec{\mathbf{r}}', t - \left|\vec{\mathbf{r}} - \vec{\mathbf{r}'}\right|\right)}{\left|\vec{\mathbf{r}} - \vec{\mathbf{r}'}\right|}\, dv'$$

$$\vec{\mathbf{A}}(\vec{\mathbf{r}}, t) = \frac{\mu_0}{4\pi} \int \frac{\vec{\mathbf{J}}\left(\vec{\mathbf{r}}', t - \left|\vec{\mathbf{r}} - \vec{\mathbf{r}'}\right|\right)}{\left|\vec{\mathbf{r}} - \vec{\mathbf{r}'}\right|}\, dv'$$

are called *retarded potentials* because the charge and current densities are evaluated at points in time before t. In other words, the potentials at time t are thought to be produced by charges and currents some distance away at a time before t.

Advanced potentials

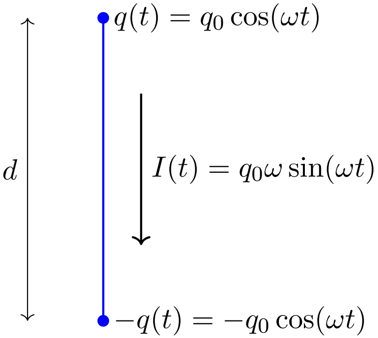
The solutions

$$V(\vec{\mathbf{r}}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho\left(\vec{\mathbf{r}}', t + \left|\vec{\mathbf{r}} - \vec{\mathbf{r}'}\right|\right)}{\left|\vec{\mathbf{r}} - \vec{\mathbf{r}'}\right|}\, dv'$$

$$\vec{\mathbf{A}}(\vec{\mathbf{r}}, t) = \frac{\mu_0}{4\pi} \int \frac{\vec{\mathbf{J}}\left(\vec{\mathbf{r}}', t + \left|\vec{\mathbf{r}} - \vec{\mathbf{r}'}\right|\right)}{\left|\vec{\mathbf{r}} - \vec{\mathbf{r}'}\right|}\, dv'$$

are called *advanced potentials* because the charge and current densities are evaluated at points in time after t. Since it seems impossible that the motion of charges and currents in the future should determine the potentials now, most people ignore the advanced potentials. Nevertheless, they are perfectly good solutions to the Maxwell equations. Perhaps some day someone will find a use for them or a different interpretation of them.

The Electric Dipole Radiator



Retarded scalar potential for the electric dipole radiator

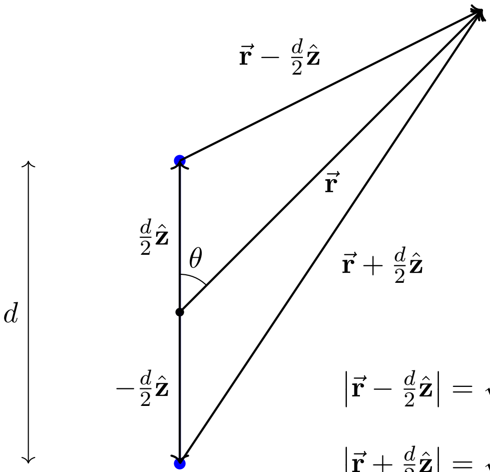
Using the retarded potential

$$V(\vec{\mathbf{r}}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho\left(\vec{\mathbf{r}}', t - \left|\vec{\mathbf{r}} - \vec{\mathbf{r}'}\right|\right)}{\left|\vec{\mathbf{r}} - \vec{\mathbf{r}'}\right|} \, dv'$$

and assuming that our oscillating charges are point charges, we have

$$\begin{aligned} V(\vec{\mathbf{r}}, t) &= \frac{1}{4\pi\epsilon_0} \left[\frac{q\left(t - \left|\vec{\mathbf{r}} - \frac{d}{2}\hat{\mathbf{z}}\right|\right)}{\left|\vec{\mathbf{r}} - \frac{d}{2}\hat{\mathbf{z}}\right|} - \frac{q\left(t - \left|\vec{\mathbf{r}} + \frac{d}{2}\hat{\mathbf{z}}\right|\right)}{\left|\vec{\mathbf{r}} + \frac{d}{2}\hat{\mathbf{z}}\right|} \right] \\ &= \frac{1}{4\pi\epsilon_0} \left\{ \frac{q_0 \cos\left[\omega\left(t - \left|\vec{\mathbf{r}} - \frac{d}{2}\hat{\mathbf{z}}\right|\right)\right]}{\left|\vec{\mathbf{r}} - \frac{d}{2}\hat{\mathbf{z}}\right|} - \frac{q_0 \cos\left[\omega\left(t - \left|\vec{\mathbf{r}} + \frac{d}{2}\hat{\mathbf{z}}\right|\right)\right]}{\left|\vec{\mathbf{r}} + \frac{d}{2}\hat{\mathbf{z}}\right|} \right\} \end{aligned}$$

Geometry of the electric dipole radiator



$$\left|\vec{\mathbf{r}} - \tfrac{d}{2}\hat{\mathbf{z}}\right| = \sqrt{r^2 + (d/2)^2 - rd\cos\theta}$$

$$\left|\vec{\mathbf{r}} + \tfrac{d}{2}\hat{\mathbf{z}}\right| = \sqrt{r^2 + (d/2)^2 + rd\cos\theta}$$

Approximation 1: $d \ll r$

$$
\begin{aligned}
\left|\vec{\mathbf{r}} \pm \frac{d}{2}\hat{\mathbf{z}}\right| &= \sqrt{r^2 + (d/2)^2 \pm rd\cos\theta} \\
&= r\sqrt{1 \pm \frac{d}{r}\cos\theta + \left(\frac{d}{2r}\right)^2} \\
&\approx r\left(1 \pm \frac{d}{2r}\cos\theta\right) \\
&= r \pm \frac{d}{2}\cos\theta
\end{aligned}
$$

Approximation 2: $d \ll \lambda$

$\lambda\omega = 2\pi$, so $d \ll \lambda$ is equivalent to $\omega d \ll 1$

$$\left|\vec{\mathbf{r}} \pm \frac{d}{2}\hat{\mathbf{z}}\right| \approx r \pm \frac{d}{2}\cos\theta$$

$$\cos\left[\omega\left(t - \left|\vec{\mathbf{r}} \pm \frac{d}{2}\hat{\mathbf{z}}\right|\right)\right] \approx \cos\left[\omega\left(t - r \mp \frac{d}{2}\cos\theta\right)\right]$$
$$= \cos[\omega(t-r)]\cos\left(\frac{\omega d}{2}\cos\theta\right) \pm \sin[\omega(t-r)]\sin\left(\frac{\omega d}{2}\cos\theta\right)$$
$$\approx \cos[\omega(t-r)] \pm \sin[\omega(t-r)]\frac{\omega d}{2}\cos\theta$$

Approximation 1 again: $d \ll r$

$$\begin{aligned}
\left|\vec{\mathbf{r}} \pm \frac{d}{2}\hat{\mathbf{z}}\right|^{-1} &= \left[r^2 + (d/2)^2 \pm rd\cos\theta\right]^{-1/2} \\
&= \frac{1}{r}\left[1 \pm \frac{d}{r}\cos\theta + \left(\frac{d}{2r}\right)^2\right]^{-1/2} \\
&\approx \frac{1}{r}\left(1 \mp \frac{d}{2r}\cos\theta\right)
\end{aligned}$$

Exercise: Plug these approximations into the potential and simplify.

$$V(\vec{\mathbf{r}},t) = \frac{1}{4\pi\epsilon_0}\left\{\frac{q_0\cos\left[\omega\left(t-\left|\vec{\mathbf{r}}-\frac{d}{2}\hat{\mathbf{z}}\right|\right)\right]}{\left|\vec{\mathbf{r}}-\frac{d}{2}\hat{\mathbf{z}}\right|} - \frac{q_0\cos\left[\omega\left(t-\left|\vec{\mathbf{r}}+\frac{d}{2}\hat{\mathbf{z}}\right|\right)\right]}{\left|\vec{\mathbf{r}}+\frac{d}{2}\hat{\mathbf{z}}\right|}\right\}$$

$$\cos\left[\omega\left(t-\left|\vec{\mathbf{r}}\pm\frac{d}{2}\hat{\mathbf{z}}\right|\right)\right] \approx \cos[\omega(t-r)] \pm \sin[\omega(t-r)]\frac{\omega d}{2}\cos\theta$$

$$\left|\vec{\mathbf{r}}\pm\frac{d}{2}\hat{\mathbf{z}}\right|^{-1} \approx \frac{1}{r}\left(1\mp\frac{d}{2r}\cos\theta\right)$$

Approximation 3: $\lambda \ll r$

$$V(\vec{\mathbf{r}}, t) = \frac{q_0 \cos\theta}{4\pi\epsilon_0 r}\left\{\frac{d}{r}\cos[\omega(t-r)] - \omega d \sin[\omega(t-r)]\right\}$$

- $\lambda\omega = 2\pi$, so $\lambda \ll r$ is equivalent to $\frac{1}{r} \ll \omega$

$$V(\vec{\mathbf{r}}, t) = \frac{q_0 d\omega \cos\theta}{4\pi\epsilon_0 r}\sin[\omega(r-t)]$$

- V is a spherical wave
- We are interested in the case $d \ll \lambda \ll r$. Is there a more systematic way to impose this condition, instead of using three approximations at various points in the derivation?

Retarded vector potential for the electric dipole radiator

Using the retarded potential

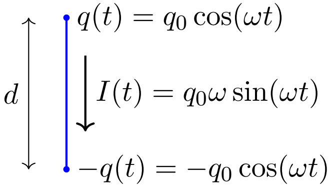
$$\vec{\mathbf{A}}(\vec{\mathbf{r}}, t) = \frac{\mu_0}{4\pi} \int \frac{\vec{\mathbf{J}}\left(\vec{\mathbf{r}}', t - \left|\vec{\mathbf{r}} - \vec{\mathbf{r}'}\right|\right)}{\left|\vec{\mathbf{r}} - \vec{\mathbf{r}'}\right|} \, dv'$$

and assuming that our oscillating current is a line charge, we have

$$\begin{aligned}\vec{\mathbf{A}}(\vec{\mathbf{r}}, t) &= \frac{\mu_0}{4\pi} \int_{-d/2}^{d/2} \frac{I(t - |\vec{\mathbf{r}} - z\hat{\mathbf{z}}|)\hat{\mathbf{z}}}{|\vec{\mathbf{r}} - z\hat{\mathbf{z}}|} \, dz \\ &\approx \frac{\mu_0 q_0 d\omega}{4\pi r} \sin[\omega(r - t)]\hat{\mathbf{z}}\end{aligned}$$

- Another spherical wave

Summary of potentials for the Electric Dipole Radiator



For the electric dipole radiator, under the conditions $d \ll \lambda \ll r$, we found the retarded potentials.

$$V(\vec{\mathbf{r}}, t) = \frac{q_0 d \omega \cos\theta}{4\pi\epsilon_0 r} \sin[\omega(r - t)]$$

$$\vec{\mathbf{A}}(\vec{\mathbf{r}}, t) = \frac{\mu_0 q_0 d \omega}{4\pi r} \sin[\omega(r - t)]\hat{\mathbf{z}}$$

Exercise: Compute the electric and magnetic fields from V and $\vec{\mathbf{A}}$.

$$V(\vec{\mathbf{r}}, t) = \frac{q_0 d\omega \cos\theta}{4\pi\epsilon_0 r} \sin[\omega(r - t)]$$

$$\vec{\mathbf{A}}(\vec{\mathbf{r}}, t) = \frac{\mu_0 q_0 d\omega}{4\pi r} \sin[\omega(r - t)]\hat{\mathbf{z}}$$

$$\vec{\mathbf{E}} = -\vec{\nabla} V - \frac{\partial \vec{\mathbf{A}}}{\partial t}$$

$$\vec{\mathbf{B}} = \vec{\nabla} \times \vec{\mathbf{A}}$$

Electric field for the electric dipole radiator

$$\begin{aligned}
\vec{\mathbf{E}} &= -\vec{\nabla} V - \frac{\partial \vec{\mathbf{A}}}{\partial t} \\
&\approx -\frac{q_0 d\omega^2 \cos\theta}{4\pi\epsilon_0 r}\cos[\omega(r-t)]\hat{\mathbf{r}} + \frac{\mu_0 q_0 d\omega^2}{4\pi r}\cos[\omega(r-t)]\hat{\mathbf{z}} \\
&= \frac{\mu_0 q_0 d\omega^2}{4\pi r}\cos[\omega(r-t)](-\cos\theta\hat{\mathbf{r}} + \cos\theta\hat{\mathbf{r}} - \sin\theta\hat{\boldsymbol{\theta}}) \\
&= -\frac{\mu_0 q_0 d\omega^2 \sin\theta}{4\pi r}\cos[\omega(r-t)]\hat{\boldsymbol{\theta}}
\end{aligned}$$

- $\vec{\mathbf{E}}$ is a spherical wave polarized in the $\hat{\boldsymbol{\theta}}$ direction.
- The $\sin\theta$ means maximum radiation at the equator, hardly any at the poles.
- The $1/r$ dependence is what makes this radiation.

At a sphere of any size around the radiator, the time-averaged energy is the same.

$$\mathcal{E} = \frac{1}{2}\left(\epsilon_0 \vec{\mathbf{E}} \cdot \vec{\mathbf{E}} + \frac{1}{\mu_0}\vec{\mathbf{B}} \cdot \vec{\mathbf{B}}\right)$$

- Since $E \propto \frac{1}{r}$, the energy density $\mathcal{E} \propto \frac{1}{r^2}$, so when we integrate that over the surface of a sphere, we will get some constant, independent of the radius of the sphere.
- The $1/r$ dependence of the electric field is what entitles this particular electric field to qualify as radiation. The energy present a distance R from the radiator will still be present at a distance of $2R$, or $10R$, or $10^6 R$.
- The electric field of a stationary charge, which is proportional to $1/r^2$, does not qualify as radiation. In this case, the energy density is proportional to $1/r^4$, so the total energy a distance R from the radiator decreases as $1/r^2$.

Magnetic field for the electric dipole radiator

$$\vec{\mathbf{A}}(\vec{\mathbf{r}}, t) = \frac{\mu_0 q_0 d\omega}{4\pi r} \sin[\omega(r-t)](\cos\theta \hat{\mathbf{r}} - \sin\theta \hat{\boldsymbol{\theta}})$$

$$\begin{aligned} \vec{\mathbf{B}} &= \vec{\nabla} \times \vec{\mathbf{A}} \\ &= -\frac{\mu_0 q_0 d\omega^2 \sin\theta}{4\pi r} \cos[\omega(r-t)]\hat{\boldsymbol{\phi}} \end{aligned}$$

- $\vec{\mathbf{B}}$ is a spherical wave polarized in the $\hat{\phi}$ direction.
- The $\sin\theta$ means maximum radiation at the equator, hardly any at the poles.
- The $1/r$ dependence is what makes this radiation.

Energy density of the electric dipole radiation field

$$\vec{\mathbf{E}} = -\frac{\mu_0 q_0 d\omega^2 \sin\theta}{4\pi r}\cos[\omega(r-t)]\hat{\boldsymbol{\theta}}$$
$$\vec{\mathbf{B}} = -\frac{\mu_0 q_0 d\omega^2 \sin\theta}{4\pi r}\cos[\omega(r-t)]\hat{\boldsymbol{\phi}}$$

$$\begin{aligned}\mathcal{E} &= \frac{1}{2}\left(\epsilon_0 \vec{\mathbf{E}}\cdot\vec{\mathbf{E}} + \frac{1}{\mu_0}\vec{\mathbf{B}}\cdot\vec{\mathbf{B}}\right)\\ &= \frac{\mu_0 q_0^2 d^2\omega^4 \sin^2\theta}{16\pi^2 r^2}\cos^2[\omega(r-t)]\end{aligned}$$

The time-averaged energy density is

$$\langle\mathcal{E}\rangle = \frac{\mu_0 q_0^2 d^2\omega^4 \sin^2\theta}{32\pi^2 r^2}.$$

Momentum density of the electric dipole radiation field

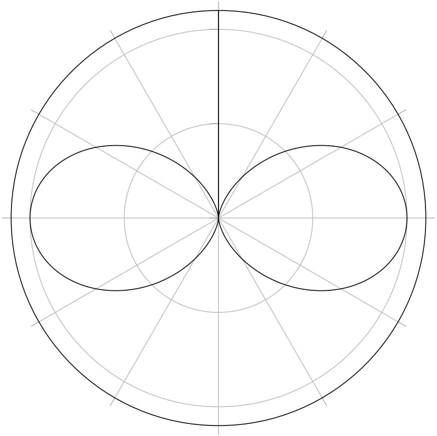
$$\vec{\mathbf{E}} = -\frac{\mu_0 q_0 d\omega^2 \sin\theta}{4\pi r}\cos[\omega(r-t)]\hat{\boldsymbol{\theta}}$$
$$\vec{\mathbf{B}} = -\frac{\mu_0 q_0 d\omega^2 \sin\theta}{4\pi r}\cos[\omega(r-t)]\hat{\boldsymbol{\phi}}$$

- When $c = 1$, momentum density is the same as Poynting vector $\vec{\mathbf{S}}$.

$$\begin{aligned}\vec{\mathbf{S}} &= \frac{1}{\mu_0}\vec{\mathbf{E}}\times\vec{\mathbf{B}} \\ &= \frac{\mu_0 q_0^2 d^2 \omega^4 \sin^2\theta}{16\pi^2 r^2}\cos^2[\omega(r-t)]\hat{\mathbf{r}}\end{aligned}$$

Electric dipole radiation field

$$\left\langle \left| \vec{\mathbf{S}} \right| \right\rangle = \frac{\mu_0 q_0^2 d^2 \omega^4 \sin^2\theta}{32\pi^2 r^2}$$



Power radiated through a sphere of radius R

$$
\begin{aligned}
P &= \int \vec{\mathbf{S}} \cdot d\vec{\mathbf{a}} \\
&= \int_0^{\pi} \int_0^{2\pi} \frac{\mu_0 q_0^2 d^2 \omega^4 \sin^2\theta}{16\pi^2 r^2} \cos^2[\omega(r-t)] r^2 \sin\theta \; d\phi \; d\theta \\
&= \frac{\mu_0 q_0^2 d^2 \omega^4}{16\pi^2} 2\pi \cos^2[\omega(R-t)] \int_0^{\pi} \sin^3\theta \; d\theta \\
&= \frac{\mu_0 q_0^2 d^2 \omega^4}{6\pi} \cos^2[\omega(R-t)]
\end{aligned}
$$

Time-averaged power radiated is

$$\langle P \rangle = \frac{\mu_0 q_0^2 d^2 \omega^4}{12\pi}.$$

- Note that the result doesn't depend upon R.
- The strong dependence on ω is why the sky is blue.

Summary for the Electric Dipole Radiator

Under the conditions $d \ll \lambda \ll r$:

$$V(\vec{\mathbf{r}}, t) = \frac{q_0 d\omega \cos\theta}{4\pi\epsilon_0 r} \sin[\omega(r - t)]$$

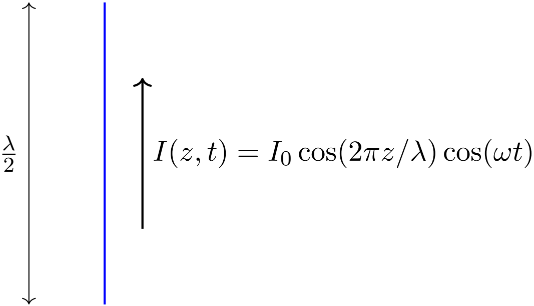
$$\vec{\mathbf{A}}(\vec{\mathbf{r}}, t) = \frac{\mu_0 q_0 d\omega}{4\pi r} \sin[\omega(r - t)]\hat{\mathbf{z}}$$

$$\vec{\mathbf{E}} = -\frac{\mu_0 q_0 d\omega^2 \sin\theta}{4\pi r} \cos[\omega(r - t)]\hat{\boldsymbol{\theta}}$$

$$\vec{\mathbf{B}} = -\frac{\mu_0 q_0 d\omega^2 \sin\theta}{4\pi r} \cos[\omega(r - t)]\hat{\boldsymbol{\phi}}$$

$$\vec{\mathbf{S}} = \frac{\mu_0 q_0^2 d^2 \omega^4 \sin^2\theta}{16\pi^2 r^2} \cos^2[\omega(r - t)]\hat{\mathbf{r}} \qquad \langle P \rangle = \frac{\mu_0 q_0^2 d^2 \omega^4}{12\pi}$$

The Half-Wavelength Dipole Antenna



$$\lambda\omega = 2\pi$$

$$\frac{\partial \lambda_c}{\partial t} + \frac{\partial I}{\partial z} = 0$$

Charge density for the half-wavelength dipole antenna

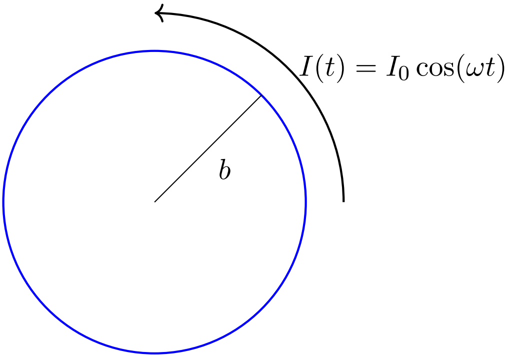
$$\lambda\omega = 2\pi$$

$$\frac{\partial \lambda_c}{\partial t} + \frac{\partial I}{\partial z} = 0$$

$$\lambda_c(z,t) = I_0 \sin(2\pi z/\lambda)\sin(\omega t)$$

$$I(z,t) = I_0 \cos(2\pi z/\lambda)\cos(\omega t)$$

The Magnetic Dipole Radiator



Retarded scalar potential for the magnetic dipole radiator

Using the retarded potential

$$V(\vec{\mathbf{r}}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho\left(\vec{\mathbf{r}'}, t - \left|\vec{\mathbf{r}} - \vec{\mathbf{r}'}\right|\right)}{\left|\vec{\mathbf{r}} - \vec{\mathbf{r}'}\right|}\, dv'$$

and realizing that the loop is uncharged, we have

$$\rho(\vec{\mathbf{r}}, t) = 0$$

and so

$$V(\vec{\mathbf{r}}, t) = 0.$$

Retarded vector potential for the magnetic dipole radiator

Using the retarded potential

$$\vec{\mathbf{A}}(\vec{\mathbf{r}}, t) = \frac{\mu_0}{4\pi} \int \frac{\vec{\mathbf{J}}\left(\vec{\mathbf{r}}', t - |\vec{\mathbf{r}} - \vec{\mathbf{r}}'|\right)}{|\vec{\mathbf{r}} - \vec{\mathbf{r}}'|} \, dv'$$

and assuming that our oscillating current is a line charge, we have

$$\vec{\mathbf{r}}' = b(\cos\phi' \hat{\mathbf{x}} + \sin\phi' \hat{\mathbf{y}})$$

$$\vec{\mathbf{A}}(\vec{\mathbf{r}}, t) = \frac{\mu_0}{4\pi} \int_0^{2\pi} \frac{I(t - |\vec{\mathbf{r}} - b\cos\phi'\hat{\mathbf{x}} - b\sin\phi'\hat{\mathbf{y}}|)\hat{\boldsymbol{\phi}}}{|\vec{\mathbf{r}} - b\cos\phi'\hat{\mathbf{x}} - b\sin\phi'\hat{\mathbf{y}}|} b \, d\phi'$$

- We should work on simplifying $|\vec{\mathbf{r}} - b\cos\phi'\hat{\mathbf{x}} - b\sin\phi'\hat{\mathbf{y}}|$.

Simplifying $\vec{\mathbf{A}}$ for the magnetic dipole radiator

$$\vec{\mathbf{A}}(\vec{\mathbf{r}},t) = \frac{\mu_0}{4\pi}\int_0^{2\pi} \frac{I(t - |\vec{\mathbf{r}} - b\cos\phi'\hat{\mathbf{x}} - b\sin\phi'\hat{\mathbf{y}}|)\hat{\boldsymbol{\phi}}}{|\vec{\mathbf{r}} - b\cos\phi'\hat{\mathbf{x}} - b\sin\phi'\hat{\mathbf{y}}|} b\ d\phi'$$

$$\vec{\mathbf{r}} = r\sin\theta\cos\phi\hat{\mathbf{x}} + r\sin\theta\sin\phi\hat{\mathbf{y}} + r\cos\theta\hat{\mathbf{z}}$$

- Exercise: Simplify the expression $|\vec{\mathbf{r}} - b\cos\phi'\hat{\mathbf{x}} - b\sin\phi'\hat{\mathbf{y}}|$.

Solution

$$\left|\vec{\mathbf{r}} - b\cos\phi'\hat{\mathbf{x}} - b\sin\phi'\hat{\mathbf{y}}\right| = \sqrt{r^2 + b^2 - 2rb\sin\theta\cos(\phi - \phi')}$$

Knowing that $b \ll r$:

- Find an approximate expression for $|\vec{\mathbf{r}} - b\cos\phi'\hat{\mathbf{x}} - b\sin\phi'\hat{\mathbf{y}}|$.
- Find an approximate expression for $|\vec{\mathbf{r}} - b\cos\phi'\hat{\mathbf{x}} - b\sin\phi'\hat{\mathbf{y}}|^{-1}$.

Approximate expressions

$$
\begin{aligned}
\left|\vec{\mathbf{r}} - \vec{\mathbf{r}}'\right| &= \sqrt{r^2 + b^2 - 2rb\sin\theta\cos(\phi - \phi')} \\
&= r\sqrt{1 - 2\frac{b}{r}\sin\theta\cos(\phi - \phi') + \left(\frac{b}{r}\right)^2} \\
&\approx r\left(1 - \frac{b}{r}\sin\theta\cos(\phi - \phi')\right) \\
&= r - b\sin\theta\cos(\phi - \phi')
\end{aligned}
$$

$$
\begin{aligned}
\left|\vec{\mathbf{r}} - \vec{\mathbf{r}}'\right|^{-1} &= [r^2 + b^2 - 2rb\sin\theta\cos(\phi - \phi')]^{-1/2} \\
&= r^{-1}\left[1 - 2\frac{b}{r}\sin\theta\cos(\phi - \phi') + \left(\frac{b}{r}\right)^2\right]^{-1/2} \\
&\approx \frac{1}{r}\left(1 + \frac{b}{r}\sin\theta\cos(\phi - \phi')\right)
\end{aligned}
$$

Calculating $\vec{\mathbf{A}}$ for the magnetic dipole radiator

$$\vec{\mathbf{A}}(\vec{\mathbf{r}},t) = \frac{\mu_0}{4\pi}\int_0^{2\pi} \frac{I(t - |\vec{\mathbf{r}} - b\cos\phi'\hat{\mathbf{x}} - b\sin\phi'\hat{\mathbf{y}}|)\hat{\boldsymbol{\phi}}}{|\vec{\mathbf{r}} - b\cos\phi'\hat{\mathbf{x}} - b\sin\phi'\hat{\mathbf{y}}|} b\ d\phi'$$

$$= \frac{\mu_0 I_0 b}{4\pi}\int_0^{2\pi} \cos\{\omega[t - r + b\sin\theta\cos(\phi - \phi')]\}$$

$$\frac{1}{r}\left(1 + \frac{b}{r}\sin\theta\cos(\phi - \phi')\right)\hat{\boldsymbol{\phi}} d\phi'$$

$$\hat{\boldsymbol{\phi}} = -\sin\phi'\hat{\mathbf{x}} + \cos\phi'\hat{\mathbf{y}}$$

- Why ϕ' and not ϕ in this expression for $\hat{\boldsymbol{\phi}}$?

Geometry of the magnetic dipole radiator