Optics from the Maxwell Equations

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Chapter 1

EM Fields in Matter

1.1 The Fields D and H

1.1.1 Polarization

Matter is made of protons and electrons, and so when matter is exposed to an electric field, the positive and negative charges in the matter respond to the electric field.

Some materials, like metals, contain electrons that are essentially free to move around in the material. These materials are called *conductors*. The electrons in a conductor respond to an electric field by redistributing, in such a way that they create their own electric field that cancels the applied electric field. An *ideal conductor* is one in which the electric field is always zero, because electrons can respond perfectly to any applied electric field.

Our main concern in this chapter is not with conductors, but with materials that are called *dielectrics*. In a dielectric, the electrons are not free to move around the material in response to an applied electric field, but are bound to a particular atom or molecule. Nevertheless, these electrons can move a little bit in response to an applied electric field, and they will. Because the electrons in a dielectric are bound to particular atoms or molecules, dielectrics are electrical insulators.

When a dielectric is exposed to an electric field, it becomes polarized. This means that the positive charges tend to move a bit in the direction of the electric field and the negative charges tend to move a bit in the opposite direction. Each molecule in the dielectric develops a dipole moment in response to the applied electric field. We define a polarization vector $\mathbf{P}$ to be the electric dipole moment per unit volume. The polarization (also called *dielectric polarization*) is larger for applied electric fields that are larger. For a *linear material*, the polarization is directly proportional to the applied electric field. For applied electric fields that are not too high, most dielectric materials behave as linear materials.

Note that we are using the term *polarization* differently than we did with EM fields. We spoke about the polarization of EM waves by labeling them linearly polarized or circularly polarized or elliptically polarized. Our use of the word polarization in here is essentially different and distinct from that in describing the polarization of EM waves. In this chapter, the polarization $\mathbf{P}$ describes the matter, not the EM field. Here, polarization is dipole moment per unit volume.

Problem 1. What are the units for dipole moment? What are the units for the polarization $\mathbf{P}$?

The electric susceptibility χ_e of a linear material describes how large the polarization is for a given applied electric field.

$$\mathbf{P} = \chi_e \mathbf{E}$$

Because of the polarization, the electric field inside the material is not the same as the applied electric field. The electric field inside the material is the sum of the external (applied) electric field and the internal electric field generated by the polarization because of the new positions of the charges.

How much electric field is generated by this polarization? The bottom line is that the polarization creates an electric field just like that created by an electric charge density

$$\rho_b(\mathbf{r}) = -\nabla \cdot \mathbf{P}(\mathbf{r}).$$

The subscript "b" in ρ_b stands for "bound," because this charge constitutes the matter that makes up the dielectric, and it is not free to move around as it pleases.

In fact we don't really want to deal with this bound charge at all. The Maxwell equations tell us that we need to be concerned with the charge density ρ. If we have matter around (which we generally do, the darn stuff is everywhere) and matter is made up of particles with electric charge, then in principle we need to keep track of all that charge. We really don't want to do that. We want to get out of that job. We're looking for a system in which we only keep track of the free charge, which is charge that's not contained in a slab of dielectric material.

We are going to divide the total charge into two parts, a free part and a bound part.

$$\rho(\mathbf{r}) = \rho_f(\mathbf{r}) + \rho_b(\mathbf{r})$$

Let us look at Gauss's law.

$$\begin{aligned}\nabla \cdot \mathbf{E} &= \rho/\epsilon_0 \\ &= \rho_f/\epsilon_0 + \rho_b/\epsilon_0 \\ &= \rho_f/\epsilon_0 - \frac{1}{\epsilon_0}\nabla \cdot \mathbf{P}\end{aligned}$$

Multiplying through by ϵ_0 and rearranging, we have

$$\nabla \cdot (\epsilon_0 \mathbf{E} + \mathbf{P}) = \rho_f.$$

1.1.2 Electric Displacement D

Now we define a new quantity, the electric displacement **D**.

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}$$

In a dielectric, we have a macroscopic version of Gauss's law.

$$\nabla \cdot \mathbf{D} = \rho_f$$

We'll call this "Gauss's law in matter."

What is the status of this new law? Is it an improved version of Gauss's law? Not really. In one small way we could answer yes to this question, but we made a number of assumptions and approximations in deriving it.

Under what conditions should we use this new law? We should use this law when we have a linear material and we don't want to be bothered about keeping track of the charge density of the protons and electrons that make up the material. The only charge density that we worry about in this setting is the free charge density, that is any charge that is not part of what makes up the dielectric material. We should also realize that the field **D** described by Gauss's law in matter is a macroscopic average field. It has nothing to tell us about how the electric field might vary as we go from one atom in the material to the next. We just want to think of the material as a slab, not worrying about atoms, and bound charges, and stuff like that.

The *permittivity* ϵ of a material is defined by the relationship

$$\mathbf{D} = \epsilon \mathbf{E}.$$

We can view free space as a dielectric material with permittivity ϵ_0.

Problem 2. Derive a relationship between ϵ and χ_e.

Problem 3. Explain why the permittivity of a linear material must be greater than or equal to ϵ_0. What would it mean physically for a material to have a permittivity of $\epsilon = \epsilon_0$?

The *dielectric constant* K_e is defined to be the dimensionless ratio

$$K_e = \frac{\epsilon}{\epsilon_0}.$$

Problem 4. Using the divergence theorem, derive an integral form of Gauss's law in matter.

Problem 5. Consider a parallel plate capacitor filled with dielectric material with dielectric constant K_e. If the plates have free surface charge densities σ_0 and $-\sigma_0$, find $\mathbf{D}$ and $\mathbf{E}$ inside the capacitor. You may assume that $\mathbf{E} = 0$ outside the capacitor.

Problem 6. Begin with Gauss's law in matter, and take for your "matter" a slab of empty space. Derive the original Gauss's law from this. This is the one sense in which we might regard Gauss's law in matter as a generalization (and hence a "better version") of Gauss's law.

Problem 7. Consider a point charge q at the center of a dielectric sphere of radius R and permittivity ϵ. Regard the point charge as free charge. Find the electric field inside and outside of the dielectric sphere.

1.1.3 Magnetization

Matter is made of protons and electrons, and so when matter is exposed to a magnetic field, the positive and negative charges in the matter respond to the magnetic field.

How do they respond? Well, we know that a moving charged particle feels a force from a magnetic field. We also know that the proton and the electron have intrinsic magnetic dipole moments, so they will feel a torque in an applied magnetic field.

Some materials, like permanent magnets, can have a magnetic dipole moment even in the absence of an applied magnetic field. These are very interesting materials, but we will not talk about them here.

Our main concern in this chapter is with materials that respond to an applied magnetic field by developing a magnetic dipole moment. Materials that do this are called *paramagnetic* if the induced magnetic dipole moment is in the same direction as the applied magnetic field, and *diamagnetic* if the induced magnetic dipole moment is in the opposite direction from the applied magnetic field.

We define a magnetization vector $\mathbf{M}$ to be the magnetic dipole moment per unit volume. The magnetic field in a material is the sum of the applied magnetic field and the magnetic field induced by the magnetization $\mathbf{M}$. How much magnetic field is generated by this magnetization? The bottom line is that the magnetization creates a magnetic field just like that created by an electric current density $\nabla \times \mathbf{M}$. This current is a kind of bound current, because it is formed by little current loops of the electrons and protons that make up the matter.

There is another source of bound current as well. A time-varying polarization can create a bound current density that is similar to Maxwell's displacement current. The bound current density can be written

$$\mathbf{J}_b = \nabla \times \mathbf{M} + \frac{\partial \mathbf{P}}{\partial t}.$$

We don't really want to deal with this bound current density at all. The Maxwell equations tell us that we need to be concerned with the current density $\mathbf{J}$. If we have matter around and matter is made up of particles with electric charge, and that electric charge is moving, then in principle we need to keep track of all that current. We really don't want to do that. We want to get out of that job. We're looking for a system in which we only keep track of the free current, which is current that's not simply a response of a material to an applied magnetic field.

We divide the current into two parts, a free part and a bound part.

$$\mathbf{J} = \mathbf{J}_f + \mathbf{J}_b$$

Let us look at the Ampere-Maxwell law.

$$\begin{aligned}\nabla \times \mathbf{B} - \mu_0\epsilon_0\frac{\partial \mathbf{E}}{\partial t} &= \mu_0\mathbf{J} \\ &= \mu_0\mathbf{J}_f + \mu_0\nabla \times \mathbf{M} + \mu_0\frac{\partial \mathbf{P}}{\partial t}\end{aligned}$$

Dividing through by μ_0 and rearranging, we have

$$\nabla \times \left(\frac{1}{\mu_0}\mathbf{B} - \mathbf{M}\right) - \frac{\partial}{\partial t}(\epsilon_0\mathbf{E} + \mathbf{P}) = \mathbf{J}_f.$$

1.1.4 H

Now we define a new quantity, the "magnetic field intensity" **H**. Many verbal descriptions of **H** have been created over the years. Many people don't like any of the verbal descriptions that have been proposed. Most people just call this field **H**.

$$\mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M}$$

In matter, we have a macroscopic version of the Ampere-Maxwell law.

$$\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J}_f$$

We'll call this the "Ampere-Maxwell law in matter."

For a *linear material*, the magnetization is directly proportional to the applied magnetic field, and also to **H**. The magnetic susceptibility χ_m of a linear material describes how large the magnetization is for a given **H**.

$$\mathbf{M} = \chi_m\mathbf{H}$$

The *permeability* μ of a material is defined by the relationship

$$\mathbf{B} = \mu\mathbf{H}.$$

We can view free space as a material with permeability μ_0.

Problem 8. Derive a relationship between μ and χ_m.

Problem 9. Using Stokes' theorem, derive an integral form of the Ampere-Maxwell law in matter.

Problem 10. Consider a solenoid filled with material with permeability μ. Suppose the solenoid has n turns of wire per unit length, with a current I flowing through the wire. Find $\mathbf{B}$ and $\mathbf{H}$ inside the solenoid. You may assume that $\mathbf{B} = 0$ outside the solenoid.

Problem 11. Begin with the Ampere-Maxwell law in matter, and take for your "matter" a slab of empty space. Derive the original Ampere-Maxwell law from this.

Problem 12. Consider a long wire carrying current I at the center of a long cylinder of radius R and permeability μ. Regard the current as free current. Find the magnetic field inside and outside of the cylinder.

1.2 The Maxwell Equations in Matter

The Maxwell equations in matter are as follows.

$$\begin{aligned} \nabla \cdot \mathbf{D} &= \rho_f & \qquad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} &= 0 \\ \nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} &= \mathbf{J}_f & \qquad \nabla \cdot \mathbf{B} &= 0 \end{aligned}$$

Problem 13. Consider a large expanse of material with permittivity ϵ and permeability μ, but with no free charge in it. Electromagnetic waves can propagate in this material. Derive a wave equation for the electric field $\mathbf{E}$ from the Maxwell equations in matter. Looking at this wave equation, what will be different about these EM waves compared with EM waves in free space?

Problem 14. Consider again a material with permittivity ϵ and permeability μ, and no free charge in it. Write expressions for the electric and magnetic fields for the four modes of EM waves with wavevector $\mathbf{k}$ in this material.

1.3 Boundary Conditions

What happens at the interface between two materials with different permittivities and permeabilities? The Maxwell equations, both original recipe and extra crispy (in matter), can help us answer this question.

Let's consider a flat planar interface between two materials. (One of the materials could be vacuum.) We are interested in what happens to **E**, **D**, **B**, and **H** at the interface. In particular, we want to know if these quantities will be continuous across the interface. In other words, will the electric field **E**, for example, have essentially the same value on one side of the interface that it has just micrometers away on the other side?

Problem 15. Consider a planar interface between two materials. The first material has permittivity ϵ_1 and permeability μ_1. The second material has permittivity ϵ_2 and permeability μ_2. Show that if the electric field is not zero and $\epsilon_1 \neq \epsilon_2$, then **D** and **E** cannot both be continuous across the boundary.

The previous problem shows that some of our physical quantities are going to have discontinuities across a boundary between two materials. The question is which ones. The answer, as usual, is contained in the Maxwell equations.

Problem 16. Write down the integral form of each of the Maxwell equations in matter.

For the next several problems, let us consider the following setup. We have one material with permittivity ϵ_1 and permeability μ_1 filling the entire region of space $z < 0$. We have a second material with permittivity ϵ_2 and permeability μ_2 filling the region $z > 0$. The plane $z = 0$ forms the boundary between the two media.

Problem 17. Consider the integral form of Gauss's law in matter. For a Gaussian surface, choose a box with length L in the x direction, width L in the y direction, and height δ in the z direction. The box is centered on the interface, with the top of the box at $z = \delta/2$, and the bottom of the box at $z = -\delta/2$. Assume the box is small enough that the field quantities **E**, **B**, **D**, and **H** don't change significantly from one location to another *within a single material*. Call the field quantities in material 1 $\mathbf{E}_1$, $\mathbf{B}_1$, $\mathbf{D}_1$, and $\mathbf{H}_1$ and those in material 2 $\mathbf{E}_2$, $\mathbf{B}_2$, $\mathbf{D}_2$, and $\mathbf{H}_2$. Write out the integral form of Gauss's law in matter using the Cartesian components of **D**. Then take the limit as $\delta \to 0$. Interpret the result.

Problem 18. Consider the integral form of the "no magnetic monopoles" law in matter. (Also called Gauss's law for magnetism in matter.) For a Gaussian surface, choose a box with length L in the x direction, width L in the y direction, and height δ in the z direction. The box is centered on the interface, with the top of the box at $z = \delta/2$, and the bottom of the box at $z = -\delta/2$. Write out this law using the Cartesian components of $\mathbf{B}$. Then take the limit as $\delta \to 0$. Interpret the result.

Problem 19. Consider the integral form of the Ampere-Maxwell law in matter. For an Amperian loop, choose a rectangle with width L in the y direction, and height δ in the z direction. The rectangle is centered on the interface, with the top of the rectangle at $z = \delta/2$, and the bottom of the rectangle at $z = -\delta/2$. Write this law using the Cartesian components of the field quantities. Then take the limit as $\delta \to 0$. Interpret the result.

Problem 20. Consider the integral form of Faraday's law in matter. For an Amperian loop, choose a rectangle with width L in the y direction, and height δ in the z direction. The rectangle is centered on the interface, with the top of the rectangle at $z = \delta/2$, and the bottom of the rectangle at $z = -\delta/2$. Write this law using the Cartesian components of the field quantities. Then take the limit as $\delta \to 0$. Interpret the result.

There are a couple more boundary conditions that we can derive, this time with the help of the *original* Maxwell equations, which still hold in matter.

Problem 21. Consider the integral form of Gauss's law (original form). For a Gaussian surface, choose a box with length L in the x direction, width L in the y direction, and height δ in the z direction. The box is centered on the interface, with the top of the box at $z = \delta/2$, and the bottom of the box at $z = -\delta/2$. Assume the box is small enough that the field quantities $\mathbf{E}$ and $\mathbf{B}$ don't change significantly from one location to another *within a single material*. Call the field quantities in material 1 $\mathbf{E}_1$ and $\mathbf{B}_1$ and those in material 2 $\mathbf{E}_2$ and $\mathbf{B}_2$. Write out the integral form of Gauss's law (original form) using the Cartesian components of $\mathbf{E}$. Then take the limit as $\delta \to 0$. Interpret the result.

Problem 22. Consider the integral form of the Ampere-Maxwell law (original version). For an Amperian loop, choose a rectangle with width L in the y direction, and height δ in the z direction. The rectangle is centered on the interface, with the top of the rectangle at $z = \delta/2$, and the bottom of the rectangle at $z = -\delta/2$. Write this law using the Cartesian components of the field quantities. Then take the limit as $\delta \to 0$. Interpret the result.

Problem 23. Summarize and organize the findings of the previous six problems by saying in words what happens to the normal component (the component perpendicular to the interface) and the tangential components (the components parallel to the interface) of each of the four field quantities.

Problem 24. Write the nicest looking equations that you can for these boundary conditions. Try to make them "coordinate independent" by avoiding explicit reference to Cartesian components.

Problem 25. (This problem due to Griffiths) The space between the plates of a parallel-plate capacitor is filled with two slabs of linear dielectric material. Each slab has thickness a, so the distance between the plates is $2a$. Slab 1 (next to the top plate) has a dielectric constant of 2, and slab 2 (next to the bottom plate) has a dielectric constant of 1.5. The free charge density on the top plate is σ and on the bottom plate is $-\sigma$.

(a) Find the electric displacement $\mathbf{D}$ in each slab.

(b) Find the electric field $\mathbf{E}$ in each slab.

(c) Find the polarization $\mathbf{P}$ in each slab.

(d) Find the potential difference between the plates.

(e) Find the location and amount of all bound charge.

(f) Now that you know all the charge (free and bound), recalculate the field in each slab, and confirm your answer to (b).

Problem 26. (This problem due to Griffiths) A sphere of linear dielectric material has embedded in it a uniform free charge density ρ. Find the potential at the center of the sphere (relative to infinity), if its radius is R and its dielectric constant is K_e.

Chapter 2

Optics

Optics is an old subject and a very large subject. For as long as people have been aware of light, they have looked for some sort of theory of how light works. I would say that today, there are four active theories of light.

1. Light is a ray.
2. Light is a wave.
3. Light is an electromagnetic wave.
4. Light is a quantum field.

These four theories of light are ordered chronologically, in terms of when they were discovered, and they are also ordered from simplest, most easy to apply, with least predictive power, to most complex, most difficult to apply, with most predictive power.

The achievement of Maxwell is theory 3 of light above, that light is an EM wave. This theory is our primary interest here. Even so, the theory has a lot to say about light and its properties, and we can only cover a small piece here.

Our main interest in this chapter is what happens to a plane wave of light that strikes a plane boundary between two media. Let's work with the following setup. We have one material (let's call it material 1) with permittivity ϵ_1 and permeability μ_1 filling the entire region of space $z < 0$. We have a second material (material 2) with permittivity ϵ_2 and permeability μ_2 filling the region $z > 0$. The plane $z = 0$ forms the boundary between the two media.

We suppose there is a plane EM wave traveling in material 1 toward the boundary with material 2. We call an EM wave that is propagating toward an interface between two materials an *incident wave*. At the interface, some of the energy and momentum of the incident wave is reflected back into material 1, away from the interface, in a wave that we call the *reflected wave*. Another portion of the energy and momentum is transmitted into material 2 in a *transmitted wave* or refracted wave.

We will derive Snell's law as well as expressions for the amount of light reflected and refracted.

2.1 Normal Incidence

First, we investigate what happens when the incident wave is traveling normal to the interface plane, that is, in the z direction. Let us choose the following expressions for our waves.

$$\begin{aligned}
\mathbf{E}_i(\mathbf{r},t) &= E_{i0}\cos(k_1 z - \omega t)\hat{\mathbf{i}} \\
\mathbf{E}_r(\mathbf{r},t) &= E_{r0}\cos(-k_1 z - \omega t)\hat{\mathbf{i}} \\
\mathbf{E}_t(\mathbf{r},t) &= E_{t0}\cos(k_2 z - \omega t)\hat{\mathbf{i}}
\end{aligned}$$

Problem 27. Write down the corresponding magnetic fields for the incident, reflected, and transmitted waves.

Problem 28. Apply the boundary condition for the tangential component of $\mathbf{E}$ to this situation. Since the boundary is located at $z = 0$, you may substitute $z = 0$ into the waves when you write out the boundary condition. The boundary condition must hold at all times, but you may substitute $t = 0$ if that makes things simpler for you. Write down a relationship that must hold between E_{i0}, E_{r0}, and E_{t0}.

Problem 29. Apply the boundary condition for the tangential component of $\mathbf{H}$ to this situation, assuming that there is no free current density at the boundary. Since the boundary is located at $z = 0$, you may substitute $z = 0$ into the waves when you write out the boundary condition. The boundary condition must hold at all times, but you may substitute $t = 0$ if that makes things simpler for you. Write down a second relationship that must hold between E_{i0}, E_{r0}, and E_{t0}.

Problem 30. Use the results of the previous two problems to solve for E_{r0} in terms of E_{i0} and to solve for E_{t0} in terms of E_{i0}. It may help to define

$$\beta = \frac{\mu_1 v_1}{\mu_2 v_2} = \frac{\mu_1 n_2}{\mu_2 n_1} = \frac{\mu_1}{\mu_2}\sqrt{\frac{\epsilon_2 \mu_2}{\epsilon_1 \mu_1}} = \sqrt{\frac{\epsilon_2 \mu_1}{\epsilon_1 \mu_2}}.$$

Express E_{r0} in terms of E_{i0} and β. Express E_{t0} in terms of E_{i0} and β.

The energy carried across a surface S per unit time is

$$P = \int_S \mathbf{S} \cdot \hat{\mathbf{n}} dA,$$

where $\hat{\mathbf{n}}$ is a unit vector perpendicular to the surface, and $\mathbf{S} = \mathbf{E} \times \mathbf{H}$ is the Poynting vector in matter. We use the symbol P because energy per unit time is power.

We wish to define a light intensity with units of power per unit area. For this purpose it is useful, as in the integral above, to define the intensity with respect to a surface with unit normal vector $\hat{\mathbf{n}}$. We define the *intensity* I at a surface to be the magnitude of the time average of $\mathbf{S} \cdot \hat{\mathbf{n}}$ at a spot on the surface.

Suppose f is a periodic function of time t with period T. The time-averaged value of f is

$$\langle f \rangle = \frac{1}{T} \int_0^T f(t)\, dt.$$

We use angle brackets to indicate the time average.

We define the intensity to be

$$I = \left| \langle \mathbf{S} \cdot \hat{\mathbf{n}} \rangle \right|.$$

Problem 31. Find the time average of the function $f(t) = A \cos \omega t$, where A and ω are constants. Does the answer make sense?

Problem 32. Find the time average of the function $f(t) = A \cos^2 \omega t$, where A and ω are constants. Does the answer make sense?

Problem 33. Find the intensities of the incident, reflected, and transmitted waves. What relationship do they satisfy? What is the physical interpretation of this relationship?

The reflection coefficient R is the ratio of reflected light intensity to incident light intensity. The transmission coefficient T is the ratio of transmitted light intensity to incident light intensity.

Problem 34. Write expressions for the reflection and transmission coefficients in terms of β. What relationship do they satisfy?

Problem 35. When we wrote the expression

$$\mathbf{E}_i(\mathbf{r}, t) = E_{i0} \cos(k_1 z - \omega t)\hat{\mathbf{i}}$$

for the incident wave, we were actually choosing one of the four modes of the EM field that propagate in direction $\mathbf{k} = k_1 \hat{\mathbf{k}}$. (Remember, in our notation $\hat{\mathbf{k}}$ is a unit vector in the z direction, and not necessarily in the direction of $\mathbf{k}$.) Write down incident waves for the electric and magnetic fields for the other three modes. What would happen if we used one of these other modes as our incident wave?

2.2 Oblique Incidence

Next, we investigate what happens when the incident wave hits the interface plane at an angle. Let us choose the following expressions for our waves.

$$\mathbf{E}_i(\mathbf{r}, t) = \mathbf{E}_{i0} \cos(\mathbf{k}_i \cdot \mathbf{r} - \omega_i t) \tag{2.1}$$

$$\mathbf{E}_r(\mathbf{r}, t) = \mathbf{E}_{r0} \cos(\mathbf{k}_r \cdot \mathbf{r} - \omega_r t) \tag{2.2}$$

$$\mathbf{E}_t(\mathbf{r}, t) = \mathbf{E}_{t0} \cos(\mathbf{k}_t \cdot \mathbf{r} - \omega_t t) \tag{2.3}$$

Problem 36. Write down the corresponding magnetic fields for the incident, reflected, and transmitted waves.

Problem 37. Given that there is no free charge or current at the surface, write the boundary conditions for each physical quantity that remains continuous over the boundary. Do not set $t = 0$ as we did in the case of normal incidence. (There should be four boundary conditions that lead to equations.)

2.2.1 A Frequency Lemma

We now present a mathematical result that will be useful for some of the upcoming problems. It says roughly that if you add two cosine functions together and get another cosine, then the frequencies of all three cosine functions must be the same.

Frequency Lemma. *If $A \neq 0$, $B \neq 0$, $C \neq 0$, $a > 0$, $b > 0$, $c > 0$, and*

$$A \cos ax + B \cos bx = C \cos cx \tag{2.4}$$

for all x, then $A + B = C$ and $a = b = c$.

Proof. Set $x = 0$ in (2.4) to obtain

$$A + B = C. \tag{2.5}$$

Taking two derivatives of (2.4) gives

$$-a^2 A \cos ax - b^2 B \cos bx = -c^2 C \cos cx. \tag{2.6}$$

Negating and setting $x = 0$ in (2.6) gives

$$a^2 A + b^2 B = c^2 C. \tag{2.7}$$

Taking two derivatives of (2.6) gives

$$a^4 A \cos ax + b^4 B \cos bx = c^4 C \cos cx. \tag{2.8}$$

Setting $x = 0$ in (2.8) gives

$$a^4 A + b^4 B = c^4 C. \tag{2.9}$$

Multiplying (2.5) by (2.9) gives

$$a^4 A^2 + a^4 AB + b^4 AB + b^4 B^2 = c^4 C^2. \tag{2.10}$$

Squaring equation (2.7) gives

$$a^4 A^2 + 2a^2 b^2 AB + b^4 B^2 = c^4 C^2. \tag{2.11}$$

Subtracting (2.11) from (2.10) gives

$$a^4 AB - 2a^2 b^2 AB + b^4 AB = 0 \tag{2.12}$$

or

$$(a^2 - b^2)^2 AB = 0. \tag{2.13}$$

Since $A \neq 0$ and $B \neq 0$, we have $a^2 = b^2$. Since $a > 0$ and $b > 0$, we have $a = b$.

Plug $b = a$ into (2.7) to obtain

$$a^2 C = c^2 C.$$

Since $C \neq 0$, we have $a^2 = c^2$. Since $a > 0$ and $c > 0$, we have $a = c$. This completes the proof that $a = b = c$. □

Problem 38. Using the boundary conditions and the frequency lemma, show that $\omega_i = \omega_r = \omega_t$ in equations (2.1)–(2.3). Since the frequencies are all the same, we will use ω from now on to refer to all of them.

Problem 39. What is the relationship between the wavenumbers k_i, k_r, and k_t (the wavenumber is the magnitude of the wavevector).

Problem 40. Apply the boundary condition for the tangential component of $\mathbf{E}$ to this situation.

Problem 41. Show that the x-components of $\mathbf{k}_i$, $\mathbf{k}_r$, and $\mathbf{k}_t$ must all be the same.

Problem 42. Must the y-components of $\mathbf{k}_i$, $\mathbf{k}_r$, and $\mathbf{k}_t$ all be the same? Why or why not? Must the z-components of $\mathbf{k}_i$, $\mathbf{k}_r$, and $\mathbf{k}_t$ all be the same? Why or why not?

At this point, let's reorient our x and y axes to make our lives simpler. Let's choose the direction of our x axis so that $\mathbf{k}_i$ lies in the xz plane.

Problem 43. Explain why $\mathbf{k}_r$ and $\mathbf{k}_t$ must also lie in the xz plane.

Under these conditions, the xz plane is called the *plane of incidence*. The *angle of incidence*, θ_i, is the angle between the vector $\mathbf{k}_i$ and the z-axis. The *angle of reflection*, θ_r, is the angle between the vector $\mathbf{k}_r$ and the z-axis. The *angle of refraction*, θ_t, is the angle between the vector $\mathbf{k}_t$ and the z-axis. All three of these angles are between 0 and 90 degrees.

Problem 44. Make a picture of the plane of incidence, showing the vectors $\mathbf{k}_i$, $\mathbf{k}_r$, $\mathbf{k}_t$, and the angles θ_i, θ_r, and θ_t.

Problem 45. Show that the angle of incidence equals the angle of reflection.

$$\theta_i = \theta_r$$

(Hint: consider what you have shown about the relationship between the components of the wavevectors and the magnitudes of the wavevectors.)

Problem 46. Prove Snell's law.

$$n_1 \sin\theta_i = n_2 \sin\theta_t$$

(Hint: consider what you have shown about the relationship between the components of the wavevectors and the magnitudes of the wavevectors.)

Our next job is to relate the amplitudes of the reflected and transmitted waves to the amplitude of the incident wave, just as we did in the case of normal incidence.

Let us choose the following expressions for our waves.

$$\begin{aligned}
\mathbf{E}_i(\mathbf{r},t) &= E_{i0}\cos(\mathbf{k}_i\cdot\mathbf{r}-\omega t)\hat{\mathbf{j}}\\
\mathbf{E}_r(\mathbf{r},t) &= E_{r0}\cos(\mathbf{k}_r\cdot\mathbf{r}-\omega t)\hat{\mathbf{j}}\\
\mathbf{E}_t(\mathbf{r},t) &= E_{t0}\cos(\mathbf{k}_t\cdot\mathbf{r}-\omega t)\hat{\mathbf{j}}
\end{aligned}$$

Problem 47. Write down the corresponding magnetic fields for the incident, reflected, and transmitted waves.

Problem 48. Make a picture of the plane of incidence, including the boundary between the two materials; the three wavevectors; the three angles θ_i, θ_r, and θ_t; and the directions of the electric and magnetic fields for each of the three waves.

Problem 49. Apply the boundary condition for the tangential component of $\mathbf{E}$ to this situation. Write down a relationship that must hold between E_{i0}, E_{r0}, and E_{t0}.

Problem 50. Apply the boundary condition for the tangential component of $\mathbf{H}$ to this situation. Write down a second relationship that must hold between E_{i0}, E_{r0}, and E_{t0}. Express your results in terms of β defined earlier and

$$\alpha = \frac{\cos\theta_t}{\cos\theta_i}.$$

What is this α and this β that we have defined? They are just a shorthand notation to save us some writing, but we should remember two things. First, if we know the material parameters ϵ_1, μ_1, ϵ_2, and μ_2, then we know β. So β is just an important combination of material parameters. Second, if we know the material parameters and the angle of incidence, then we know α.

Problem 51. Write an expression for α in terms of the material parameters and the angle of incidence.

Problem 52. Apply the boundary condition for the normal component of $\mathbf{B}$ to this situation. Write down a third relationship that must hold between E_{i0}, E_{r0}, and E_{t0}.

Problem 53. Are these three relationships independent? Explain.

Problem 54. Use the results of the previous problems to solve for E_{r0} in terms of E_{i0} and to solve for E_{t0} in terms of E_{i0}.

Problem 55. Find the intensities of the incident, reflected, and transmitted waves. What relationship do they satisfy? What is the physical interpretation of this relationship?

Problem 56. Write expressions for the reflection and transmission coefficients in terms of α and β. What relationship do they satisfy?

Problem 57. Consider an air-glass interface with $\mu_1 = \mu_2 = \mu_0$, $n_1 = 1$, and $n_2 = 1.5$. Make a plot of the reflection coefficient as a function of incidence angle.

Problem 58. Consider a glass-air interface with $\mu_1 = \mu_2 = \mu_0$, $n_1 = 1.5$, and $n_2 = 1$. Make a plot of the reflection coefficient as a function of incidence angle.

Problem 59. When we wrote the expression

$$\mathbf{E}_i(\mathbf{r}, t) = E_{i0} \cos(\mathbf{k}_i \cdot \mathbf{r} - \omega t)\hat{\mathbf{j}}$$

for the incident wave, we were actually choosing one of the four modes of the EM field that propagate in direction $\mathbf{k}_i$. Write down incident waves for the electric and magnetic fields for the other three modes. What would happen if we used one of these other modes as our incident wave?

The previous analysis, in which the electric field is polarized perpendicular to the plane of incidence, is called TE (transverse electric) polarization. The polarization in which the magnetic field points perpendicular to the plane of incidence is called TM (transverse magnetic) polarization.

Problem 60. Write down the electric and magnetic fields for the incident, reflected, and transmitted waves for the case of TM polarization.

Problem 61. Make a picture of the plane of incidence, including the boundary between the two materials; the three wavevectors; the three angles θ_i, θ_r, and θ_t; and the directions of the electric and magnetic fields for each of the three waves.

Problem 62. Apply the boundary condition for the tangential component of $\mathbf{E}$ to this situation. Write down a relationship that must hold between E_{i0}, E_{r0}, and E_{t0}.

Problem 63. Apply the boundary condition for the tangential component of $\mathbf{H}$ to this situation. Write down a second relationship that must hold between E_{i0}, E_{r0}, and E_{t0}.

Problem 64. Apply the boundary condition for the normal component of $\mathbf{D}$ to this situation. Write down a third relationship that must hold between E_{i0}, E_{r0}, and E_{t0}.

Problem 65. Are these three relationships independent? Explain.

Problem 66. Use the results of the previous problems to solve for E_{r0} in terms of E_{i0} and to solve for E_{t0} in terms of E_{i0}. It may help to express your results in terms of α and β.

Problem 67. Find the intensities of the incident, reflected, and transmitted waves. What relationship do they satisfy? What is the physical interpretation of this relationship?

Problem 68. Write expressions for the reflection and transmission coefficients in terms of α and β. What relationship do they satisfy?

Problem 69. Consider an air-glass interface with $\mu_1 = \mu_2 = \mu_0$, $n_1 = 1$, and $n_2 = 1.5$. Make a plot of the reflection coefficient for a TM polarized wave as a function of incidence angle.

Problem 70. Consider a glass-air interface with $\mu_1 = \mu_2 = \mu_0$, $n_1 = 1.5$, and $n_2 = 1$. Make a plot of the reflection coefficient for a TM polarized wave as a function of incidence angle.

Problem 71. We have material 1 with index of refraction 1 filling the entire region of space $z < 0$. We have material 2 with index of refraction 4/3 filling the region $z > 0$. The plane $z = 0$ forms the boundary between the two media, and has no free surface charge density. Both materials have permeability μ_0. A plane EM wave in material 1 has electric field

$$\mathbf{E}_i = (100\ \text{V/m})\cos\left[\frac{(12x + 9z)}{1500\ \text{nm}} - (3 \times 10^{15}\ \text{rad/s})t\right]\hat{\mathbf{y}}.$$

(a) What is the wavelength of this wave?

(b) Write a magnetic field for this wave.

(c) As this EM wave hits the boundary, there will be a reflected wave. Write the electric and magnetic fields for the reflected wave.

(d) As this EM wave hits the boundary, there will be a transmitted wave. Write the electric and magnetic fields for the transmitted wave.

(e) Confirm that the tangential component of $\mathbf{E}$ is continuous across the boundary.

(f) Confirm that the tangential component of $\mathbf{H}$ is continuous across the boundary.

(g) Confirm that the normal component of $\mathbf{B}$ is continuous across the boundary.

(h) Confirm that the incident intensity is equal to the reflected intensity plus the transmitted intensity.

Problem 72. For one of the polarizations (TE or TM), there is an angle (called Brewster's angle) at which there is no reflected light. Which polarization has this feature? Find an expression for Brewster's angle in terms of the material parameters n_1 and n_2 in the simplified case in which $\mu_1 = \mu_2$.

Problem 73. Get some polarizing film from the lab and show me Brewster's angle.

Problem 74. Another interesting angle is the critical angle for total internal reflection. Under what conditions does this occur? Find an expression for this angle. What goes to zero when the angle of incidence is greater than this angle? Show how this idea is contained in the expressions we have derived. Do both polarizations have the same critical angle?