Complex Numbers

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1 Introduction

Complex numbers are pairs of real numbers. If x and y are real numbers, then the pair (x, y) can be regarded as a complex number. Usually, we write this complex number as

$$x + yi.$$

We call the real number x the *real part* of the complex number and the real number y the *imaginary part* of the complex number.

> **Example:** The numbers $\frac{1}{2}$ and $-\frac{\sqrt{3}}{2}$ are real numbers, so the pair $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ can be regarded as a complex number, which we will typically write as
>
> $$\frac{1}{2} - \frac{\sqrt{3}}{2}i.$$
>
> The complex number $\frac{1}{2} - \frac{\sqrt{3}}{2}i$ can be written in other ways.
>
> $$\frac{1}{2} - \frac{\sqrt{3}}{2}i = \frac{1}{2} - i\frac{\sqrt{3}}{2} = -\frac{\sqrt{3}}{2}i + \frac{1}{2} = -i\frac{\sqrt{3}}{2} + \frac{1}{2}$$

2 Real Numbers

We regard the real numbers as a subset of the complex numbers. Real numbers are those complex numbers that have zero for their imaginary part. Instead of writing $4 + 0i$ for such a number, we will just write 4. In this way, every real number is a complex number.

Complex numbers can be added, subtracted, multiplied, and divided.

3 Addition of Complex Numbers

If $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$ are complex numbers, then the sum $z_1 + z_2$ is defined to be

$$z_1 + z_2 := (x_1 + x_2) + (y_1 + y_2)i.$$

The symbol ":=" means "is defined to be". To add two complex numbers, we just add the real parts and add the imaginary parts.

> **Example:** Let $z_1 = -3 - 4i$ and $z_2 = 5i - 7$. Then the sum is $z_1 + z_2 = -10 + i$.

4 Subtraction of Complex Numbers

If $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$ are complex numbers, then the difference $z_1 - z_2$ is defined to be

$$z_1 - z_2 := (x_1 - x_2) + (y_1 - y_2)i.$$

To subtract two complex numbers, we just subtract the real parts and subtract the imaginary parts.

> **Example:** Let $z_1 = -3 - 4i$ and $z_2 = 5i - 7$. Then the difference is $z_1 - z_2 = 4 - 9i$.

5 Multiplication of Complex Numbers

If $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$ are complex numbers, then the product $z_1 z_2$ is defined to be

$$z_1 z_2 := (x_1 x_2 - y_1 y_2) + (x_1 y_2 + y_1 x_2)i.$$

The best way to think about this is to imagine that you are distributing the product term by term, and using the idea that $i^2 = -1$.

$$\begin{aligned}(x_1 + y_1 i)(x_2 + y_2 i) &= x_1 x_2 + x_1 y_2 i + y_1 i x_2 + y_1 i y_2 i \\ &= x_1 x_2 + x_1 y_2 i + y_1 x_2 i - y_1 y_2 \\ &= (x_1 x_2 - y_1 y_2) + (x_1 y_2 + y_1 x_2)i\end{aligned}$$

6 The Complex Conjugate

The *complex conjugate* of a complex number $z = x + yi$ is defined to be the complex number

$$z^* := x - yi.$$

Mathematicians usually write $\overline{z}$ for the complex conjugate of z, while physicists usually write z^*. We will use the physicists' convention in these notes. You can think of the complex conjugate as being formed by replacing every i in a complex number by $-i$.

If you multiply any complex number by its complex conjugate, you get a real number.

$$(x + yi)(x - yi) = x^2 + y^2$$

7 Division of Complex Numbers

If $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$ are complex numbers, and $z_2 \neq 0$, then we can form a quotient. We can simplify the quotient by multiplying the numerator and denominator by the complex conjugate of the denominator.

$$\begin{aligned}\frac{z_1}{z_2} &= \frac{x_1 + y_1 i}{x_2 + y_2 i} \\ &= \frac{x_1 + y_1 i}{x_2 + y_2 i}\left(\frac{x_2 - y_2 i}{x_2 - y_2 i}\right) \\ &= \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + \frac{-x_1 y_2 + y_1 x_2}{x_2^2 + y_2^2} i\end{aligned}$$

8 The Complex Plane

Since complex numbers are pairs of real numbers, it makes sense to think of them as points on a plane. By convention, we use the horizontal axis to represent the real part of a complex number, and the vertical axis to represent the imaginary part.

9 Magnitude of a Complex Number

The *magnitude* of a complex number is the distance from the origin on the complex plane. Therefore, the magnitude of a complex number must always be a nonnegative real number. The magnitude will be zero if and only if the complex number itself is zero. If $z = x + yi$ is a complex number, we write $|z|$ to denote the magnitude of z, and

$$|z| = \sqrt{x^2 + y^2}$$

from the Pythagorean theorem.

10 Exponential Function

If $w = u + vi$ is a complex number (with u and v real numbers), we define

$$e^w = e^{u+vi} = e^u(\cos v + i \sin v)$$

where v is understood as an angle in radians. This is really just a fancy version of the Euler formula, which holds if θ is real.

$$e^{i\theta} = \cos\theta + i \sin\theta$$

11 Polar Form of a Complex Number

Consider the complex number $z = Re^{i\theta}$, where R and θ are real numbers, and R is nonnegative. Using the Euler formula, we can write this as

$$z = Re^{i\theta} = R(\cos\theta + i\sin\theta) = R\cos\theta + iR\sin\theta$$

This complex number has real part $R\cos\theta$ and imaginary part $R\sin\theta$. This is just the point in the complex plane with polar coordinates R and θ! Notice that R is the magnitude of the complex number. The angle θ is called the *argument* of the complex number z.

Every complex number z can be written in *Cartesian form*

$$z = x + yi$$

with x and y real, as well as *polar form*

$$z = Re^{i\theta}$$

with R and θ real. The relationship between these two forms for writing complex numbers is the same as the relationship between Cartesian and polar coordinates.

$$x = R\cos\theta \qquad R = \sqrt{x^2 + y^2}$$
$$y = R\sin\theta \qquad \theta = \arctan\frac{y}{x}$$

12 Multiplication in Polar Form

Multiplication of complex numbers is especially easy to do if they are written in polar form.

$$R_1 e^{i\theta_1} R_2 e^{i\theta_2} = R_1 R_2 e^{i(\theta_1+\theta_2)}$$

To form the product of complex numbers in polar form you multiply the magnitudes and add the arguments.

13 Problems

Try to avoid decimal computation in the following exercises. (For example, write $\sqrt{2}$ instead of 1.41.....)

Problem 1 *Simplify the following and express in Cartesian form.*

1. $(4e^{i\pi/2})(4e^{i\pi})$
2. $4e^{i\pi/2} + 4e^{i\pi}$
3. $3e^{i\pi/4} + 4e^{i3\pi/4}$
4. $(1+i)(\sqrt{3}-i)$

Problem 2 *Simplify the following and express in polar form.*

1. $(4e^{i\pi/2})(4e^{i\pi})$
2. $4e^{i\pi/2} + 4e^{i\pi}$
3. $(1+i)(\sqrt{3}-i)$
4. $(e^{i\pi/6})(e^{i\pi/9}) + (2e^{i3\pi/2})(e^{-i2\pi/9})$

Problem 3 *Find* i^*i, $(-i)^*(-i)$, $|i|^2$, $|-i|^2$, $|i|$, $|-i|$.

Problem 4 *Let* $z = 3 + 4i$. *Find* z^*, z^*z, $|z|$, z^2, *and* $1/z$.

Problem 5 *Is it always true that* $|z|^2 = z^*z$ *? Show that it's true or find a counterexample. (Hint: If you want to try to show that it's true, write* $z = x + iy$ *with* x *and* y *real.)*

Problem 6 *Find two different values for* z *so that* $z^2 = i$. *Express these numbers in both Cartesian and polar form. Plot them on the complex plane.*

Problem 7 *Show, on the complex plane, all of the complex numbers with magnitude 1.*

Problem 8 *Simplify* $|e^{i\omega t} + e^{-i\omega t}|^2$, *where* ω *and* t *are real numbers. Is the result a real number? If so, write it in a way that does not have an* i *in it.*