Magnetic Vector Potential

Scott N. Walck

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Modern Electromagnetic Theory

- The Maxwell Equations

$$\vec{\nabla} \times \vec{B} - \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{J} \qquad \vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon_0} \rho$$

$$\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \qquad \vec{\nabla} \cdot \vec{B} = 0$$

- The Lorentz Force Law

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

- Electric potential does not show up in the Maxwell equations or the Lorentz force law. It's role is somehow "behind the scenes".

Theorem

Let $\vec{B}$ be a vector field. If

$$\vec{\nabla} \cdot \vec{B} = 0$$

then there exists a vector field $\vec{A}$ such that

$$\vec{B} = \vec{\nabla} \times \vec{A}.$$

- A magnetic field is always the curl of some vector field.
- (Most vector fields are not the curl of any vector field.)

Magnetic Vector Potential

Since the magnetic field is divergenceless

$$\vec{\nabla} \cdot \vec{B} = 0$$

there must be a vector field $\vec{A}$ such that

$$\vec{B} = \vec{\nabla} \times \vec{A}.$$

We call the vector field $\vec{A}$ the *magnetic vector potential*, or just the *vector potential*.

- This magnetic vector potential is not as pleasant as the electric potential because it's a vector instead of a scalar.

The magnetic field does not uniquely determine a vector potential.

If A_0 and B_0 are constants, the following vector potentials all produce the same magnetic field.

- $\vec{A} = B_0 y \hat{z}$
- $\vec{A} = -B_0 z \hat{y}$
- $\vec{A} = \frac{1}{2} B_0 y \hat{z} - \frac{1}{2} B_0 z \hat{y}$
- $\vec{A} = B_0 y \hat{z} + A_0 \hat{x}$
- $\vec{A} = B_0 y \hat{z} + A_0 \hat{y}$
- $\vec{A} = B_0 y \hat{z} + A_0 \hat{z}$

What magnetic field is it?

Potentials

$$\vec{E} = -\vec{\nabla} V \qquad \vec{B} = \vec{\nabla} \times \vec{A}$$

- The equation for $\vec{E}$ automatically satisfies Kirchhoff's law: $\vec{\nabla} \times \vec{E} = 0$.
- The equation for $\vec{B}$ automatically satisfies the no-magnetic-monopoles law: $\vec{\nabla} \cdot \vec{B} = 0$.
- Bad news: when we leave statics and start letting things change in time, we'll expand Kirchhoff's law into Faraday's law and the equation for $\vec{E}$ will need to be modified to satisfy Faraday's law rather than Kirchhoff's law.
- Good news: the equation for $\vec{B}$ is in its final form.

Ampere's Law and Vector Potential

- Start with Ampere's law.

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

- Replace $\vec{B}$ with $\vec{\nabla} \times \vec{A}$ and use the identity $\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A}$.

$$\vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A} = \mu_0 \vec{J}$$

- If we agree to use an $\vec{A}$ with no divergence, then we get a simpler expression.

$$\nabla^2 \vec{A} = -\mu_0 \vec{J}$$