The Vector Line Integral

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Eight types of integrals

name of integral	notation	dim.	requires	
scalar line integral	$\int_C f\, d\ell$	1D	scalar field	curve
vector line integral	$\int_C \vec{F}\, d\ell$	1D	vector field	curve
dotted line integral	$\int_C \vec{F} \cdot d\vec{\ell}$	1D	vector field	curve
scalar surface integral	$\int_S f\, da$	2D	scalar field	surface
vector surface integral	$\int_S \vec{F}\, da$	2D	vector field	surface
flux integral	$\int_S \vec{F} \cdot d\vec{a}$	2D	vector field	surface
scalar volume integral	$\int_V f\, dv$	3D	scalar field	volume
vector volume integral	$\int_V \vec{F}\, dv$	3D	vector field	volume

Integrals in Griffiths section 1.3.1

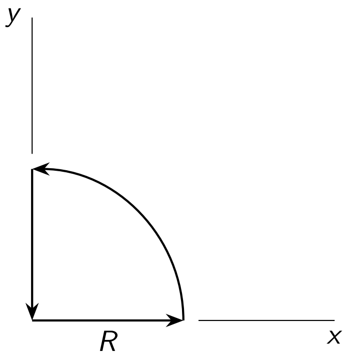
notation	inputs		output	In 1.3.1?
$\int_C f\, d\ell$	scalar field	curve	scalar	no
$\int_C \vec{F}\, d\ell$	vector field	curve	vector	no
$\int_C \vec{F} \cdot d\vec{\ell}$	vector field	curve	scalar	yes
$\int_S f\, da$	scalar field	surface	scalar	no
$\int_S \vec{F}\, da$	vector field	surface	vector	no
$\int_S \vec{F} \cdot d\vec{a}$	vector field	surface	scalar	yes
$\int_V f\, dv$	scalar field	volume	scalar	yes
$\int_V \vec{F}\, dv$	vector field	volume	vector	yes

Example

Find the vector line integral $\int_C \vec{F}\, d\ell$ for the vector field

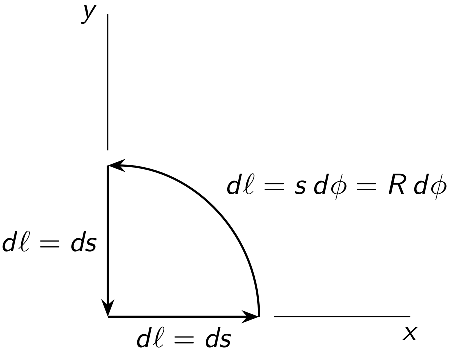
$$\vec{F}(s, \phi, z) = s^2 \cos\phi \hat{s} + s^2 \sin\phi \hat{\phi}$$

over the closed path C shown below.



Line elements

$$d\vec{\ell} = ds\,\hat{s} + s\,d\phi\,\hat{\phi} + dz\,\hat{z}$$



$$\int_C \vec{F}\,d\ell = \int_{C_1} \vec{F}\,d\ell + \int_{C_2} \vec{F}\,d\ell + \int_{C_3} \vec{F}\,d\ell$$

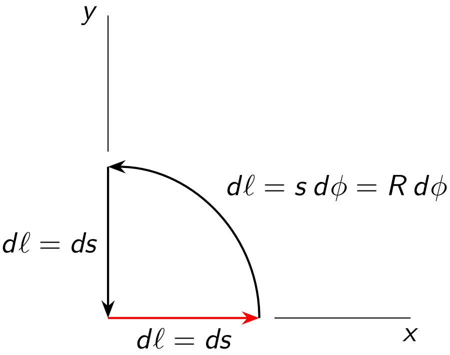
Use Cartesian unit vectors under the integral.

$$\begin{aligned}
\vec{F} &= s^2 \cos\phi \hat{s} + s^2 \sin\phi \hat{\phi} \\
&= s^2 \cos\phi(\cos\phi \hat{x} + \sin\phi \hat{y}) + s^2 \sin\phi(-\sin\phi \hat{x} + \cos\phi \hat{y}) \\
&= s^2 \cos 2\phi \hat{x} + s^2 \sin 2\phi \hat{y}
\end{aligned}$$

Here, we have used the following trigonometric identities.

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$\sin 2\theta = 2\sin\theta\cos\theta$$

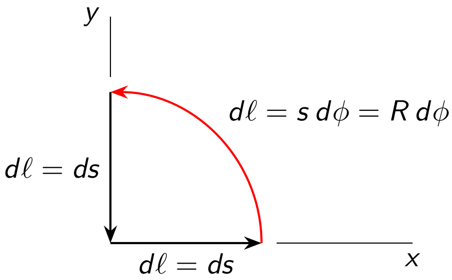
First portion



$$\int_{C_1} \vec{F}\,d\ell = \int_0^R (s^2 \cos 2\phi \hat{x} + s^2 \sin 2\phi \hat{y})\,ds$$

$$= \hat{x} \int_0^R s^2\,ds = \frac{R^3}{3}\hat{x}$$

Second portion



$$\int_{C_2} \vec{F}\,d\ell = \int_0^{\pi/2} (s^2 \cos 2\phi \hat{x} + s^2 \sin 2\phi \hat{y}) s\,d\phi$$

$$= R^3 \int_0^{\pi/2} (\cos 2\phi \hat{x} + \sin 2\phi \hat{y})\,d\phi$$

$$= R^3 \hat{x} \int_0^{\pi/2} \cos 2\phi\,d\phi + R^3 \hat{y} \int_0^{\pi/2} \sin 2\phi\,d\phi$$

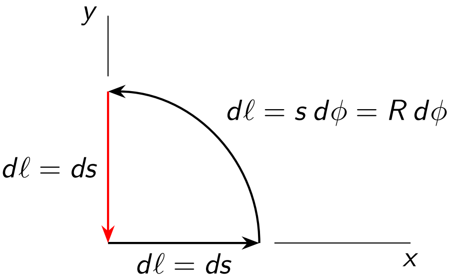
Second portion, continued

$$\int_0^{\pi/2} \cos 2\phi \, d\phi = \left[\frac{\sin 2\phi}{2}\right]_0^{\pi/2} = 0$$

$$\int_0^{\pi/2} \sin 2\phi \, d\phi = \left[-\frac{\cos 2\phi}{2}\right]_0^{\pi/2} = \left[\frac{\cos 2\phi}{2}\right]_{\pi/2}^{0} = \frac{1-(-1)}{2} = 1$$

$$\int_{C_2} \vec{F} \, d\ell = R^3 \hat{y}$$

Third portion



$$\int_{C_3} \vec{F}\, d\ell = \int_R^0 (s^2 \cos 2\phi \hat{x} + s^2 \sin 2\phi \hat{y})\, ds$$

$$= -\hat{x} \int_R^0 s^2\, ds = \frac{R^3}{3} \hat{x}$$

In total we have the following result.

$$\int_C \vec{F}\, d\ell = \frac{2}{3} R^3 \hat{x} + R^3 \hat{y}$$

Your turn

Find the vector line integral $\int_C \vec{F}\, d\ell$ for the vector field

$$\vec{F} = \frac{1}{r^2}\hat{r}$$

over the straight-line path from $(x, y, z) = (1, 0, 0)$ to $(x, y, z) = (1, 4, 0)$.

An easier problem

Find the vector line integral $\int_C \vec{F}\, d\ell$ for the vector field

$$\vec{F} = \frac{1}{r^2}\hat{r}$$

over the straight-line path from $(x, y, z) = (1, 0, 0)$ to $(x, y, z) = (4, 0, 0)$.