Vector Algebra

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A little review of regular (non-vector) algebra

Suppose a, b, and c are numbers.

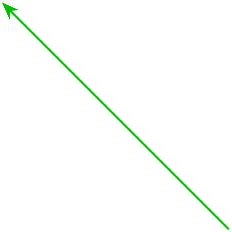
- What does the commutative law for multiplication say?
- What does the associative law for multiplication say?
- What does the distributive law say?

Vectors in 3-dimensional space

- Picture a vector as an arrow.
- By an arrow, we mean a straight line segment with an arrow head on one end.
- A vector has a magnitude and (almost always) a direction.

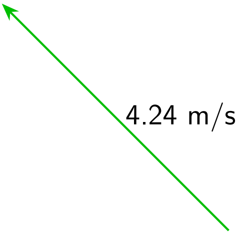
The *direction* of the vector points with the arrowhead.

- A vector pointing northwest



The *magnitude* of the vector is the length of the arrow.

- A vector with magnitude 4.24 m/s.



There is a *zero vector*.

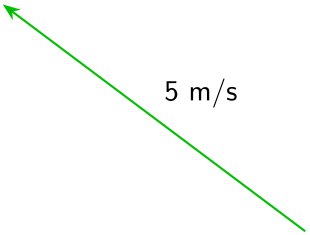
- The magnitude of the zero vector is zero.
- If a vector has a magnitude of zero, it must be the zero vector.

  $$\text{zero vector} \Leftrightarrow \text{vector with zero magnitude}$$

- The zero vector does not have a direction. (It's the only vector that has no direction.)
- We can draw the zero vector as a point, or not draw it at all.

All nonzero vectors have a magnitude and a direction.

- The magnitude of a nonzero vector is a positive number, along with units.
- We draw a vector as an arrow.
- The length of the arrow represents the magnitude of the vector.
- The direction of the arrow represents the direction of the vector.



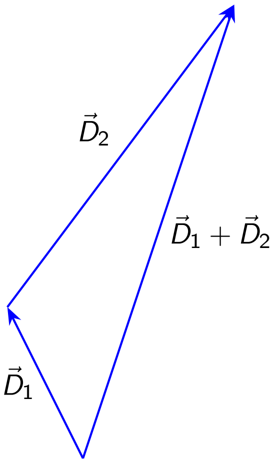
Scalars versus vectors

- What are some physical quantities or physical properties represented as vectors?
- What are some physical quantities or physical properties represented as scalars?

What can we do with vectors?

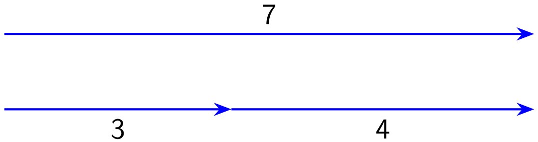
- We can add two vectors to produce a new vector.
- We can subtract two vectors to produce a new vector.
- We can scale a vector by a number to produce a new vector.
- We can multiply two vectors in two ways (dot product and cross product).

To add vectors, put them tip-to-tail

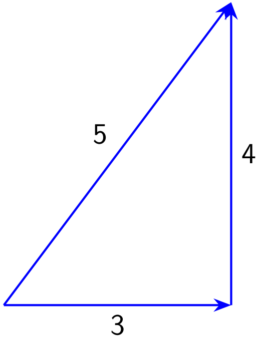


- Vector sum points from tail of first vector to tip of last vector

Vectors can do everything that numbers can do

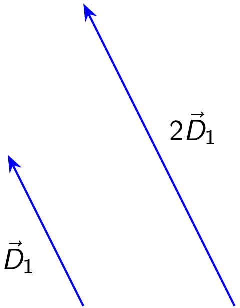


Vectors can do more than numbers can do

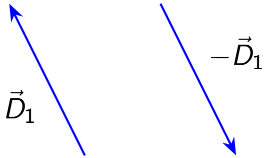


- A vector with magnitude 3 plus a vector with magnitude 4 can produce a vector with magnitude 5.
- Magnitude of the sum (5) $\neq$ the sum of the magnitudes (7).

Vectors can scale

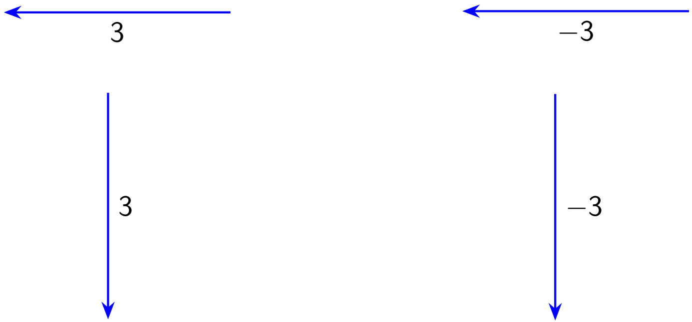


To negate a vector, flip its direction



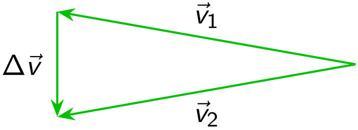
Vectors have magnitude and direction

Write this | Don't write this



- You can label a vector with its magnitude.
- The magnitude of a vector is never negative.

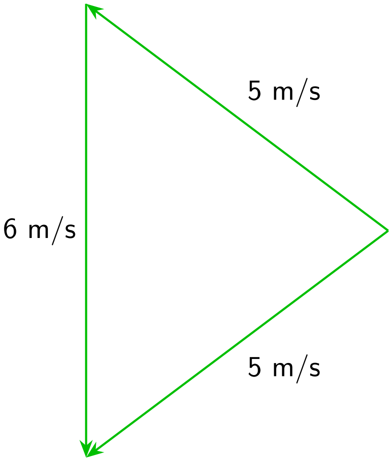
To subtract vectors, put them tail-to-tail



$$\Delta\vec{v} = \vec{v}_2 - \vec{v}_1$$

$$\vec{v}_2 = \vec{v}_1 + \Delta\vec{v}$$

Magnitude of difference $\neq$ difference of magnitudes



- Magnitude of difference $= 6$ m/s
- Difference of magnitudes $= 0$ m/s

Vectors vs. Numbers

- Numbers can be positive.
- Numbers can be negative.
- Numbers can increase (over time).
- Numbers can decrease (over time).
- Vectors cannot be positive.
- Vectors cannot be negative.
- Vectors cannot increase (over time).
- Vectors cannot decrease (over time).
- Numbers have an order. If a and b are numbers on the number line, then either $a > b$, $a < b$, or $a = b$.
- Vectors do not have an order.

Use an arrow to denote a vector.

- $\vec{A}$ is a vector.
- A is not a vector.
- Books often use boldface for a vector.
- Mathematicians don't do anything special for a vector.
- Physicists like to have a syntactic way to show a vector.

Vector components

If you have a coordinate system (you can always choose a coordinate system if you don't have one), you can express a vector in terms of its coordinate components.

$$\vec{A} = A_x \hat{x} + A_y \hat{y} + A_z \hat{z}$$

Magnitude of a vector

If
$$\vec{A} = A_x \hat{x} + A_y \hat{y} + A_z \hat{z}$$
then the magnitude of $\vec{A}$ is
$$\left|\vec{A}\right| = \sqrt{A_x^2 + A_y^2 + A_z^2}.$$

We will use the convention that if $\vec{A}$ is a vector, then A is the magnitude of the vector.
$$A = \left|\vec{A}\right|$$

So, A and $\vec{A}$ both have meaning, and they mean different things! Please use the arrow if you mean a vector, and don't use the arrow if you don't mean a vector.

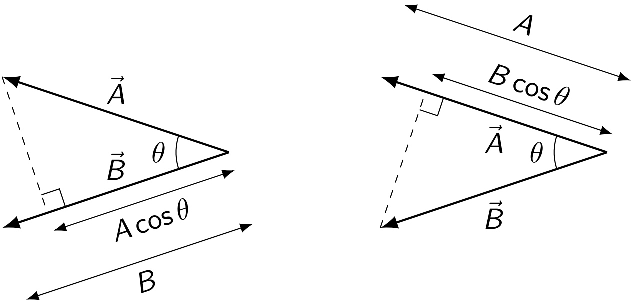
Use a hat for a unit vector.

If $\vec{A}$ is a vector, then $\hat{A}$ is a unit vector in the direction of $\vec{A}$.

$$\hat{A} = \frac{\vec{A}}{\left|\vec{A}\right|} = \frac{\vec{A}}{A}$$

Dot Product

$$\vec{A} \cdot \vec{B} = AB\cos\theta$$



- The dot product of two vectors is a scalar.
- The dot product probes the extent to which two vectors point in the same direction.

Properties of the Dot Product

- The dot product is commutative.

$$\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$$

- The dot product is related to the magnitude of a vector.

$$\vec{A} \cdot \vec{A} = \left|\vec{A}\right|^2 = A^2 \qquad \text{so} \qquad A = \sqrt{\vec{A} \cdot \vec{A}}$$

- The dot product distributes over a vector sum.

$$\vec{C} \cdot (\vec{A} + \vec{B}) = \vec{C} \cdot \vec{A} + \vec{C} \cdot \vec{B}$$

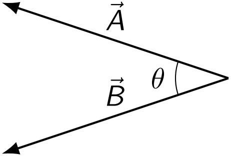
Dot products of Cartesian unit vectors

$$\hat{x} \cdot \hat{x} = 1 \qquad \hat{x} \cdot \hat{y} = 0 \qquad \hat{x} \cdot \hat{z} = 0$$
$$\hat{y} \cdot \hat{x} = 0 \qquad \hat{y} \cdot \hat{y} = 1 \qquad \hat{y} \cdot \hat{z} = 0$$
$$\hat{z} \cdot \hat{x} = 0 \qquad \hat{z} \cdot \hat{y} = 0 \qquad \hat{z} \cdot \hat{z} = 1$$

$$\begin{aligned}\vec{A} \cdot \vec{B} &= (A_x\hat{x} + A_y\hat{y} + A_z\hat{z}) \cdot (B_x\hat{x} + B_y\hat{y} + B_z\hat{z}) \\ &= A_xB_x + A_yB_y + A_zB_z\end{aligned}$$

Cross Product

$$\left|\vec{A} \times \vec{B}\right| = AB \sin\theta$$



- The vector $\vec{A} \times \vec{B}$ points out of the page.
- The vector $\vec{B} \times \vec{A}$ points into the page.
- The cross product of two vectors is a vector.

Properties of the Cross Product

- The cross product is *anti*-commutative.

$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$$

- The cross product distributes over a vector sum.

$$\vec{C} \times (\vec{A} + \vec{B}) = \vec{C} \times \vec{A} + \vec{C} \times \vec{B}$$

Cross products of Cartesian unit vectors

$$
\begin{aligned}
\hat{x} \times \hat{x} &= 0 & \hat{x} \times \hat{y} &= \hat{z} & \hat{x} \times \hat{z} &= -\hat{y} \\
\hat{y} \times \hat{x} &= -\hat{z} & \hat{y} \times \hat{y} &= 0 & \hat{y} \times \hat{z} &= \hat{x} \\
\hat{z} \times \hat{x} &= \hat{y} & \hat{z} \times \hat{y} &= -\hat{x} & \hat{z} \times \hat{z} &= 0
\end{aligned}
$$

$$
\begin{aligned}
\vec{A} \times \vec{B} &= (A_x\hat{x} + A_y\hat{y} + A_z\hat{z}) \times (B_x\hat{x} + B_y\hat{y} + B_z\hat{z}) \\
&= (A_yB_z - A_zB_y)\hat{x} + (A_zB_x - A_xB_z)\hat{y} + (A_xB_y - A_yB_x)\hat{z}
\end{aligned}
$$

Example: Find the angle between the face diagonals of a cube.

- Draw a cube.
- Choose one of the eight vertices to be the origin.
- Lay a Cartesian coordinate system on the cube.
- Identify two face diagonals. Write them as vectors.
- Take the dot product of these two vectors. By equating the geometric definition of dot product with the expression for dot product in Cartesian coordinates, we can solve for the angle between the vectors.

Position vectors

- Position is not inherently a vector, but given a coordinate system, we can form a position vector.
- The *field point* is a position where we want to know the electric field.
$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$$
- The *source point* is a position where some charge lives.
$$\vec{r'} = x'\hat{x} + y'\hat{y} + z'\hat{z}$$

Separation Vector

- The separation vector points from the source point to the field point.

$$\vec{\mathscr{r}} = \vec{r} - \vec{r'}$$