The Scalar Line Integral

Scott N. Walck

September 23, 2024

Eight types of integrals

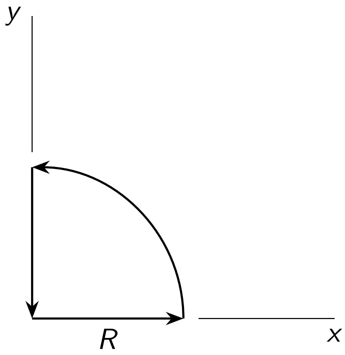
name of integral	notation	dim.	requires	
scalar line integral	$\int_C f\,d\ell$	1D	scalar field	curve
vector line integral	$\int_C \vec{F}\,d\ell$	1D	vector field	curve
dotted line integral	$\int_C \vec{F}\cdot d\vec{\ell}$	1D	vector field	curve
scalar surface integral	$\int_S f\,da$	2D	scalar field	surface
vector surface integral	$\int_S \vec{F}\,da$	2D	vector field	surface
flux integral	$\int_S \vec{F}\cdot d\vec{a}$	2D	vector field	surface
scalar volume integral	$\int_V f\,dv$	3D	scalar field	volume
vector volume integral	$\int_V \vec{F}\,dv$	3D	vector field	volume

Example

Find the scalar line integral $\int_C f \, d\ell$ for the scalar field

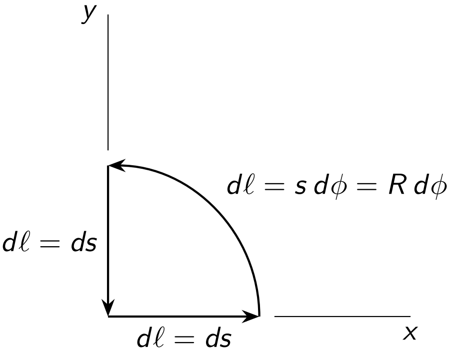
$$f(s, \phi, z) = s^2 \cos \phi$$

over the closed path C shown below.



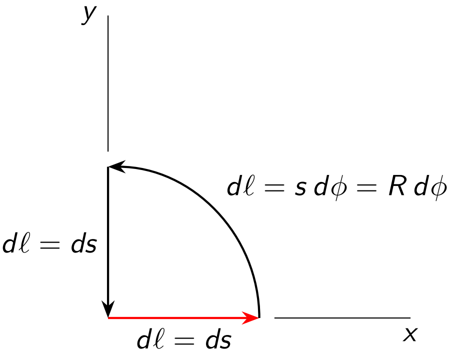
Line elements

$$d\vec{\ell} = ds\,\hat{s} + s\,d\phi\,\hat{\phi} + dz\,\hat{z}$$



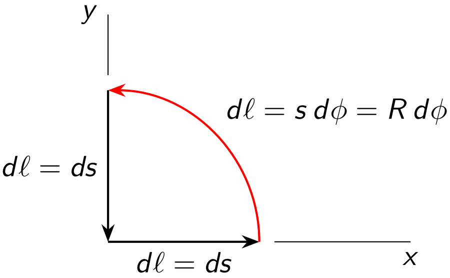
$$\int_C f\,d\ell = \int_{C_1} f\,d\ell + \int_{C_2} f\,d\ell + \int_{C_3} f\,d\ell$$

First portion



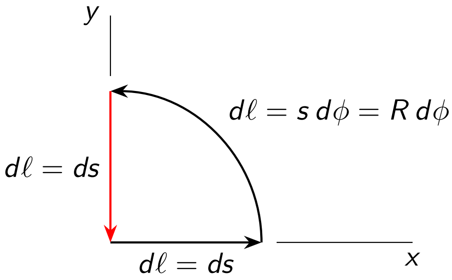
$$\int_{C_1} f\,d\ell = \int_0^R (s^2 \cos\phi)\,ds$$
$$= \int_0^R s^2\,ds = \frac{R^3}{3}$$

Second portion



$$\int_{C_2} f\,d\ell = \int_0^{\pi/2} (s^2 \cos\phi) s\,d\phi$$
$$= R^3$$

Third portion



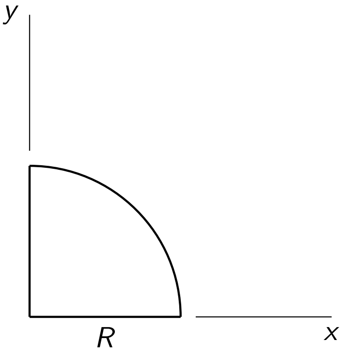
$$\int_{C_3} f\,d\ell = \int_R^0 (s^2 \cos\phi)\,ds$$
$$= 0$$

In total we have the following result.

$$\int_C f\,d\ell = \frac{1}{3}R^3 + R^3 + 0 = \frac{4}{3}R^3$$

Physics application

Find the total electric charge distributed over the closed curve shown below.



The linear charge density is not uniform, but rather distributed according to the function

$$\lambda(s, \phi, z) = \lambda_0 \left(\frac{s}{R}\right)^2 \cos\phi$$

Physics application, continued

$$\lambda(s,\phi,z) = \lambda_0\left(\frac{s}{R}\right)^2 \cos\phi$$

The total charge is

$$Q = \int dq = \int_P \lambda(\vec{r})\, d\ell = \frac{\lambda_0}{R^2}\int_P s^2 \cos\phi\, d\ell$$
$$= \frac{\lambda_0}{R^2}\frac{4}{3}R^3 = \frac{4}{3}\lambda_0 R$$