Magnetostatics

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Modern Electromagnetic Theory

- The Maxwell Equations

$$\vec{\nabla} \times \vec{B} - \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{J} \qquad \vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon_0} \rho$$

$$\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \qquad \vec{\nabla} \cdot \vec{B} = 0$$

- The Lorentz Force Law

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

Magnetostatics: Five Big Ideas

- Current Distributions
- Biot-Savart Law
- Ampere's Law
- Lorentz Force Law
- Vector Potential

In static and steady situations, the Maxwell equations decouple into two electric equations and two magnetic equations.

- The Static Maxwell Equations

$$\vec{\nabla} \times \vec{E} = 0 \qquad \vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon_0}\rho \qquad \vec{\nabla} \cdot \vec{B} = 0$$

The Biot-Savart law is the solution to Ampere's law and the no-magnetic-monopoles law.

- If

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} \qquad \vec{\nabla} \cdot \vec{B} = 0$$

- then the contribution $d\vec{B}(\vec{r})$ to the magnetic field at location $\vec{r}$ by a small current segment $d\vec{\ell'}$ located at $\vec{r'}$ carrying current I is

$$d\vec{B}(\vec{r}) = \frac{\mu_0 I}{4\pi}\, d\vec{\ell'} \times \frac{\vec{r} - \vec{r'}}{\left|\vec{r} - \vec{r'}\right|^3}.$$

Current Distributions

Current distribution	Dimensionality	Symbol	SI unit
Point current	0	not possible	
Current	1	I	A
Surface current density	2	$\vec{K}$	A/m
Volume current density	3	$\vec{J}$	A/m^2

Current through a surface S:

$$I = \int_S \vec{J} \cdot d\vec{a}$$

Biot-Savart Law

We compute the magnetic field produced by an entire wire by adding together the contributions of all the small segments.

$$\vec{B}(\vec{r}) = -\frac{\mu_0 I}{4\pi} \int_C \frac{\vec{r} - \vec{r'}}{\left|\vec{r} - \vec{r'}\right|^3} \times d\vec{\ell'}$$

Conservation of Charge

Continuity equation:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0$$

Conservation of Charge

$\frac{dQ}{dt} = 0$	true	too weak
$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0$	true	just right
$\frac{d\rho}{dt} = 0$	false	too strong

Local Conservation of Charge

Continuity equation:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0$$

Choose any volume V.

$$\int_V \frac{\partial \rho}{\partial t}\, d\tau + \int_V \vec{\nabla} \cdot \vec{J}\, d\tau = 0$$

Use divergence theorem.

$$\frac{\partial}{\partial t} \int_V \rho\, d\tau + \int_{\partial V} \vec{J} \cdot d\vec{a} = 0$$

$$\frac{d}{dt} Q_{\text{enc}} + I_{\text{out}} = 0$$

Local Conservation of Charge 2

$$\frac{d}{dt}Q_{\text{enc}} + I_{\text{out}} = 0$$

$$I_{\text{in}} = -I_{\text{out}}$$

$$\frac{d}{dt}Q_{\text{enc}} = I_{\text{in}}$$

The rate at which the charge in any volume changes is equal to the current flowing into the volume.

Local Conservation of Charge: Summary

The continuity equation

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0$$

says that in order for the charge in any region to change, current must flow through the boundary of that region.