Laplace's Equation

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October 20, 2021

Laplace's Equation

Electric potential ϕ

$$\nabla^2 \phi = 0$$

$$\phi = 320 \text{ V}$$

$$\phi = ?$$

$$\phi = 300 \text{ V}$$

Temperature T

$$\nabla^2 T = 0$$

$$T = 320 \text{ K}$$

$$T = ?$$

$$T = 300 \text{ K}$$

Solution to Laplace's Equation

Electric potential ϕ

$$\nabla^2 \phi = 0$$

$\phi = 320$ V

$\phi = 315$ V

$\phi = 310$ V

$\phi = 305$ V

$\phi = 300$ V

Temperature T

$$\nabla^2 T = 0$$

$T = 320$ K

$T = 315$ K

$T = 310$ K

$T = 305$ K

$T = 300$ K

Algebraic Solution to Laplace's Equation

$$\nabla^2 \phi = 0$$

Suppose that we know from symmetry that ϕ does not depend on x or y, but depends only on z.
Cartesian coordinates are natural to choose in this situation.

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \phi = 0$$

Since ϕ doesn't depend on x or y,

$$\frac{\partial^2 \phi}{\partial z^2} = 0$$

Algebraic Solution to Laplace's Equation, page 2

$$\frac{\partial^2 \phi}{\partial z^2} = 0$$

The function ϕ depends only on one variable, so

$$\frac{d^2 \phi}{dz^2} = 0$$

$$\frac{d\phi}{dz} = A$$

General solution:

$$\phi(z) = Az + B$$

Algebraic Solution to Laplace's Equation, page 3

$$\phi(z) = Az + B$$

Now apply boundary conditions. Suppose $\phi(L/2) = 320$ V and $\phi(-L/2) = 300$ V. At $z = L/2$,

$$320 \text{ V} = A(L/2) + B$$

At $z = -L/2$,

$$300 \text{ V} = A(-L/2) + B$$

Subtract the two equations to get $20 \text{ V} = AL$, so

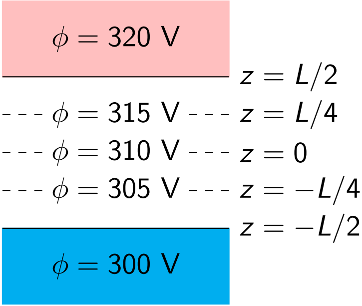
$$A = \frac{20 \text{ V}}{L} \qquad\qquad B = 310 \text{ V}$$

$$\phi(z) = (20 \text{ V})\frac{z}{L} + 310 \text{ V}$$

Compare geometric intuition with algebraic solution

Electric potential ϕ

$$\nabla^2 \phi = 0$$



$$\phi(z) = (20\text{ V})\frac{z}{L} + 310\text{ V}$$

$$\phi(L/2) = (20\text{ V})\frac{L/2}{L} + 310\text{ V} = 320\text{ V}$$

$$\phi(L/4) = (20\text{ V})\frac{L/4}{L} + 310\text{ V} = 315\text{ V}$$

$$\phi(0) = (20\text{ V})\frac{0}{L} + 310\text{ V} = 310\text{ V}$$

$$\phi(-L/4) = -\frac{1}{4}(20\text{ V}) + 310\text{ V} = 305\text{ V}$$

$$\phi(-L/2) = -\frac{1}{2}(20\text{ V}) + 310\text{ V} = 300\text{ V}$$