Integral Vector Calculus

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Three classes of integrals

Integral class	Dimensionality	Requires
Line integral	1	curve
Surface integral	2	surface
Volume integral	3	volume

Eight types of integrals

name of integral	notation	dim.	requires	
scalar line integral	$\int_C f\, d\ell$	1D	scalar field	curve
vector line integral	$\int_C \vec{F}\, d\ell$	1D	vector field	curve
dotted line integral	$\int_C \vec{F} \cdot d\vec{\ell}$	1D	vector field	curve
scalar surface integral	$\int_S f\, da$	2D	scalar field	surface
vector surface integral	$\int_S \vec{F}\, da$	2D	vector field	surface
flux integral	$\int_S \vec{F} \cdot d\vec{a}$	2D	vector field	surface
scalar volume integral	$\int_V f\, dv$	3D	scalar field	volume
vector volume integral	$\int_V \vec{F}\, dv$	3D	vector field	volume

Inputs and outputs of the vector integrals

notation	inputs		output
$\int_C f\,d\ell$	scalar field	curve	scalar
$\int_C \vec{F}\,d\ell$	vector field	curve	vector
$\int_C \vec{F}\cdot d\vec{\ell}$	vector field	curve	scalar
$\int_S f\,da$	scalar field	surface	scalar
$\int_S \vec{F}\,da$	vector field	surface	vector
$\int_S \vec{F}\cdot d\vec{a}$	vector field	surface	scalar
$\int_V f\,dv$	scalar field	volume	scalar
$\int_V \vec{F}\,dv$	vector field	volume	vector

Line integral advice

- Use the differential line element appropriate to your coordinate system. In Cartesian coordinates:

$$d\vec{\ell} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$$

- Pick *one* variable (usually x, y, or z) to be your variable of integration.
- Use knowledge of the curve to express x, y, and z in terms of your one variable. Similarly, express dx, dy, and dz in terms of your one variable.
- Choose limits for your one variable of integration from knowledge of the curve.
- Once you reduce the line integral to an ordinary integral in one variable, evaluate that integral to get the result.

Line integrals should be denoted with the two inputs they need: a function and a curve.

Better to write

$$\int_{\mathcal{P}} \vec{v} \cdot d\vec{\ell}$$

where $\mathcal{P}$ is a path from $\vec{a}$ to $\vec{b}$, rather than

$$\int_{\vec{a}}^{\vec{b}} \vec{v} \cdot d\vec{\ell}.$$

The latter is poor notation, because it implies that the integral depends only on the endpoints.

Dotted line integral example

- Vector field:

$$\vec{v} = y^2\hat{x} + 2x(y+1)\hat{y}$$

- Curve:
  (1) $(x, y, z) = (1, 1, 0) \to (2, 1, 0) \to (2, 2, 0)$
  (2) Straight line from $(x, y, z) = (1, 1, 0) \to (2, 2, 0)$
- Solution:

$$d\vec{\ell} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z} \qquad \text{(in every problem)}$$

$$\vec{v}\cdot d\vec{\ell} = y^2\,dx + 2x(y+1)\,dy \qquad \text{(in this problem)}$$

How *not* to do it

$$\vec{v} \cdot d\vec{\ell} = y^2\,dx + 2x(y+1)\,dy \qquad \text{(this is correct)}$$

$$\begin{aligned} \int \vec{v} \cdot d\vec{\ell} &= \int \left[y^2\,dx + 2x(y+1)\,dy\right] && \text{(this is dangerous)} \\ &= \int y^2\,dx + 2x \int (y+1)\,dy && \text{(this is confused)} \\ &= xy^2 + xy^2 + 2xy + C && \text{(this is wrong)} \end{aligned}$$

- The first step is dangerous because no path has been specified.
- The second step is confused because things are being organized into what look like regular integrals, but that's not what a line integral is.
- The third step is wrong, because the integrals are being treated as regular integrals.

Use the curve to reduce the integrand to *one* variable.

$$\vec{v} \cdot d\vec{\ell} = y^2\, dx + 2x(y+1)\, dy$$

- For path (i), $y = 1$, so $dy = 0$; we'll use x as our variable.

$$\vec{v} \cdot d\vec{\ell} = y^2\, dx + 2x(y+1)\, dy = dx$$

- For path (ii), $x = 2$, so $dx = 0$; we'll use y as our variable.

$$\vec{v} \cdot d\vec{\ell} = y^2\, dx + 2x(y+1)\, dy = 4(y+1)\, dy$$

- For path (2), $y = x$, so $dy = dx$; we'll use x as our variable.

$$\begin{aligned} \vec{v} \cdot d\vec{\ell} &= y^2\, dx + 2x(y+1)\, dy \\ &= x^2\, dx + 2x(x+1)\, dx = (3x^2 + 2x)\, dx \end{aligned}$$

Use the curve to reduce the integrand to *one* variable.

- Path (i): $\vec{v} \cdot d\vec{\ell} = dx$
- Path (ii): $\vec{v} \cdot d\vec{\ell} = 4(y+1)\,dy$
- Path (2): $\vec{v} \cdot d\vec{\ell} = (3x^2 + 2x)\,dx$
- Path (1) has two parts.

$$\int_{(1)} \vec{v} \cdot d\vec{\ell} = \int_{(i)} \vec{v} \cdot d\vec{\ell} + \int_{(ii)} \vec{v} \cdot d\vec{\ell}$$
$$= \int_1^2 dx + \int_1^2 4(y+1)\,dy = 11$$

- Path (2) has one part.

$$\int_{(2)} \vec{v} \cdot d\vec{\ell} = \int_1^2 (3x^2 + 2x)\,dx = 10$$

Surface integral advice

- Choose a differential surface element $d\vec{a}$ from knowledge of the surface. In Cartesian coordinates, it's usually one of the following:

$$d\vec{a} = dx\,dy\,\hat{z} \qquad d\vec{a} = dy\,dz\,\hat{x} \qquad d\vec{a} = dz\,dx\,\hat{y}$$
$$d\vec{a} = -dx\,dy\,\hat{z} \qquad d\vec{a} = -dy\,dz\,\hat{x} \qquad d\vec{a} = -dz\,dx\,\hat{y}$$

- Pick *two* variables (usually x and y, y and z, or x and z) to be your variables of integration.
- Hopefully, your third variable has a constant value across your surface. Plug in that constant value.
- Choose limits for your two variables of integration from knowledge of the surface.
- Once you reduce the surface integral to an ordinary double integral in two variables, evaluate that integral to get the result.

Volume integral advice

- Use the differential volume element $d\tau$ appropriate to your coordinate system. In Cartesian coordinates:

$$d\tau = dx\,dy\,dz$$

- We are doing a triple integral. For the innermost (first of three) integration, think of the lower and upper limits as surfaces. To describe such a surface, write the first (innermost) variable of integration in terms of the other two variables.
- For the second integration, think of the lower and upper limits as curves. To describe such a curve, write the second variable of integration in terms of the third variable.
- For the third and final integration, think of the lower and upper limits as numbers.
- Evaluate the triple integral to get a result.