Gauss's Law

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Modern Electromagnetic Theory

- The Maxwell Equations

$$\vec{\nabla} \times \vec{B} - \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{J} \qquad \vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon_0} \rho$$

$$\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \qquad \vec{\nabla} \cdot \vec{B} = 0$$

- The Lorentz Force Law

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

Gauss's Law

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon_0} \rho$$

- The divergence of electric field at a point in space is proportional to the charge density at that point.
- Where there is positive charge density, the divergence of the electric field is positive. Where there is negative charge density, the divergence of the electric field is negative.
- Electric field points away from positive charge and toward negative charge.

Flux

- The *flux* of a vector field $\vec{F}$ over a surface S is defined to be the dotted surface integral (or flux integral) of $\vec{F}$ over S.

$$\int_S \vec{F} \cdot d\vec{a}$$

- If the vector field is the electric field, we call the flux *electric flux*.

$$\Phi_E = \int_S \vec{E} \cdot d\vec{a}$$

- The surface S can be open or closed, but you must have a surface to talk about electric flux.

Deriving the integral form of Gauss's Law

- Start with the differential form of Gauss's law.

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon_0}\rho$$

- Pick any volume V. Integrate both sides of Gauss's law over this volume V.

$$\int_V (\vec{\nabla} \cdot \vec{E})\, dv = \frac{1}{\epsilon_0} \int_V \rho\, dv$$

- Use the divergence theorem to rewrite the left side. Realize that the integral on the right side is the charge contained in the volume.

$$\int_{\partial V} \vec{E} \cdot d\vec{a} = \frac{1}{\epsilon_0} Q_{\mathrm{enc}}$$

- The term on the left is the electric flux through the boundary of V.

Integral form of Gauss's Law

$$\int_{\partial V} \vec{E} \cdot d\vec{a} = \frac{1}{\epsilon_0} Q_{\mathrm{enc}}$$

- The electric flux through a closed surface is proportional to the charge enclosed by the surface.
- If a charge distribution has high symmetry, we can use the integral form of Gauss's law to find the electric field produced by the charge distribution.

Using Gauss's law to find the electric field when the charge distribution has spherical symmetry

- With spherical symmetry, ρ depends only on r, not on θ or ϕ.
- Any rotation about an axis through the origin leaves the charge distribution unchanged, and hence must leave the electric field unchanged.
- The electric field must point in the $\hat{r}$ direction: $\vec{E} = E\hat{r}$.
- The radial component E of the electric field can depend only on r, not on θ or ϕ.
- Choose a sphere of radius r as your Gaussian surface S. Notice that

$$\int_S \vec{E} \cdot d\vec{a} = E 4\pi r^2.$$

Using Gauss's law to find the electric field when the charge distribution has cylindrical symmetry

- With cylindrical symmetry, ρ depends only on s, not on ϕ or z.
- Any rotation about the symmetry axis leaves the charge distribution unchanged, and hence must leave the electric field unchanged.
- Any translation parallel to the symmetry axis leaves the charge distribution unchanged, and hence must leave the electric field unchanged.
- The electric field must point in the $\hat{s}$ direction: $\vec{E} = E\hat{s}$.
- The radial component E of the electric field can depend only on s, not on ϕ or z.
- Choose a cylinder of radius s and length L as your Gaussian surface S. Notice that

$$\int_S \vec{E} \cdot d\vec{a} = E 2\pi s L.$$

Using Gauss's law to find the electric field when the charge distribution has planar symmetry

- With planar symmetry, ρ depends only on z, not on x or y.
- Any rotation about the z axis (or any axis parallel to z) leaves the charge distribution unchanged, and hence must leave the electric field unchanged.
- Any translation in the xy plane leaves the charge distribution unchanged, and hence must leave the electric field unchanged.
- The electric field must point in the $\hat{z}$ direction: $\vec{E} = E\hat{z}$.
- The z component E of the electric field can depend only on z, not on x or y.
- Choose a box with height $2z$, width L, and length L as your Gaussian surface S. Notice that

$$\int_S \vec{E} \cdot d\vec{a} = E2L^2.$$