Fundamental Theorems of (Vector) Calculus

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October 18, 2024

Fundamental Theorem of Calculus

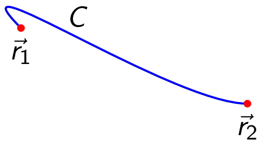
The integral of the derivative of a function is equal to the difference of the values of the function at the upper and lower limits.

Let $f : \mathbb{R} \to \mathbb{R}$ be a function.

$$\int_a^b \frac{df}{dx} dx = f(b) - f(a)$$

Gradient Theorem

The dotted line integral of the gradient of a scalar field over a curve is equal to the difference of the values of the scalar field at the end points of the curve.

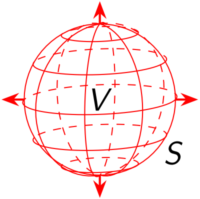


Let C be a curve from $\vec{r}_1$ to $\vec{r}_2$.
Let f be a scalar field (a scalar function of space), so that $f(\vec{r}_1)$ is a scalar.
Then $\vec{\nabla} f$ is a vector field.

$$\int_C \vec{\nabla} f \cdot d\vec{\ell} = f(\vec{r}_2) - f(\vec{r}_1)$$

Divergence Theorem

The volume integral of the divergence of a vector field is equal to the flux integral of the vector field over the boundary surface.



Let V be a volume with boundary surface S.
Let $\vec{F}$ be a vector field.

$$\int_V (\vec{\nabla} \cdot \vec{F}) d\tau = \int_S \vec{F} \cdot d\vec{a}$$

Divergence is flux per unit volume.

$$\vec{\nabla} \cdot \vec{F} = \lim_{V \to 0} \frac{\int_{\partial V} \vec{F} \cdot d\vec{a}}{\int_V dv}$$

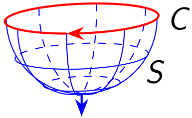
- This equation can serve as a definition of the divergence.
- It is pleasing as a definition, because it makes no reference to any particular coordinate system. It is a geometric definition.
- The practical drawback is that to use this as a definition, we would need to develop all of integral vector calculus first, before differential vector calculus. Perhaps some day I will attempt this.
- In any case, whether this equation is thought of as a definition or merely a property, knowing that divergence is flux per unit volume is conceptually helpful.

Comparison of two divergence equations

	$\vec{\nabla} \cdot \vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$	$\vec{\nabla} \cdot \vec{F} = \lim_{V \to 0} \frac{\int_{\partial V} \vec{F} \cdot d\vec{a}}{\int_V dv}$
coordinates	Cartesian	independent
calculation	easy	unpleasant
conceptual	opaque	insightful
character	algebraic	geometric

Stokes' Theorem

The flux integral of the curl of a vector field over a surface is equal to the dotted line integral of the vector field over the curve describing the boundary of the surface.



Let S be a surface with boundary curve C.
Let $\vec{F}$ be a vector field.

$$\int_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{a} = \int_C \vec{F} \cdot d\vec{\ell}$$

- The orientation of the surface S is linked to the orientation of the boundary curve C by a right-hand rule.

Notation

Let

- VF be the set of all vector fields,
- SF be the set of all scalar fields,
- Vol be the set of all volumes,
- Srf be the set of all surfaces,
- Crv be the set of all curves,
- Pts be the set of all pairs of points in space, and
- $\mathbb{R}$ be the set of all real numbers.

Boundary

- The boundary of a volume is a surface. If V is a volume, then ∂V is a (closed) surface.
- The boundary of a surface is a curve. If S is a surface, then ∂S is a (closed) curve.
- The boundary of a curve is a pair of points. If C is a curve, then ∂C is a pair of points.

Gradient Theorem

$$
\begin{array}{ccc}
SF \times Crv & \xrightarrow{\quad grad \times id \quad} & VF \times Crv \\
{\scriptstyle id \times \partial}\downarrow & & \downarrow{\scriptstyle \int} \\
SF \times Pts & \xrightarrow[\quad eval,\, subtract \quad]{} & \mathbb{R}
\end{array}
$$

$$
\begin{array}{ccc}
(f, C) & \xmapsto{\quad grad \times id \quad} & (\vec{\nabla} f, C) \\
{\scriptstyle id \times \partial}\big\downarrow & & \big\downarrow{\scriptstyle \int} \\
(f, (\vec{r}_1, \vec{r}_2)) & \xmapsto[\quad eval,\, subtract \quad]{} & f(\vec{r}_2) - f(\vec{r}_1) = \int_C (\vec{\nabla} f) \cdot d\vec{\ell}
\end{array}
$$

Stokes' Theorem

$$\begin{array}{ccc}
VF \times Srf & \xrightarrow{\;curl \times id\;} & VF \times Srf \\
\Big\downarrow {\scriptstyle id \times \partial} & & \Big\downarrow {\scriptstyle \int} \\
VF \times Crv & \xrightarrow[\;\int\;]{} & \mathbb{R}
\end{array}$$

$$\begin{array}{ccc}
(\vec{F}, S) & \xmapsto{\;curl \times id\;} & (\vec{\nabla} \times \vec{F}, S) \\
\Big\downarrow {\scriptstyle id \times \partial} & & \Big\downarrow {\scriptstyle \int} \\
(\vec{F}, \partial S) & \xmapsto[\;\int\;]{} & \int_{\partial S} \vec{F} \cdot d\vec{\ell} = \int_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{a}
\end{array}$$

Divergence Theorem

$$
\begin{array}{ccc}
VF \times Vol & \xrightarrow{\;div \times id\;} & SF \times Vol \\
\Big\downarrow {\scriptstyle id \times \partial} & & \Big\downarrow {\scriptstyle \int} \\
VF \times Srf & \xrightarrow[\;\int\;]{} & \mathbb{R}
\end{array}
$$

$$
\begin{array}{ccc}
(\vec{F}, V) & \xmapsto{\;div \times id\;} & (\vec{\nabla} \cdot \vec{F}, V) \\
\Big\downarrow {\scriptstyle id \times \partial} & & \Big\downarrow {\scriptstyle \int} \\
(\vec{F}, \partial V) & \xmapsto[\;\int\;]{} & \int_{\partial V} \vec{F} \cdot d\vec{a} = \int_V (\vec{\nabla} \cdot \vec{F}) d\tau
\end{array}
$$