Faraday's Law

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Differential Form of Faraday's Law

$$\vec{\nabla} \times \vec{\mathbf{E}} + \frac{\partial \vec{\mathbf{B}}}{\partial t} = 0$$

- In electrostatics, we had

  $$\vec{\nabla} \times \vec{\mathbf{E}} = 0.$$

  We might call this the Pre-Faraday Law or Kirchhoff's Law.
- Faraday's Law is one of the four Maxwell equations.

Connections between Electricity and Magnetism

1. Electric current produces magnetic field (Oersted's discovery, 1820).
2. Changing magnetic field produces electric field (Faraday's discovery, 1831).
3. Changing electric field produces magnetic field (Maxwell's discovery, 1865).
4. Light is a wave of electric and magnetic fields (Maxwell's discovery, 1865).

Integral form of Faraday's Law

$$\mathcal{E} = -\frac{d\Phi_B}{dt}$$

- emf around a loop is the negative rate of change of magnetic flux though a surface with the loop as boundary
- Changing magnetic flux produces an emf.

The Maxwell Equations (SI Units)

$$\vec{\nabla} \cdot \vec{\mathbf{E}} = \frac{1}{\epsilon_0}\rho$$ Gauss's Law

$$\vec{\nabla} \times \vec{\mathbf{E}} + \frac{\partial \vec{\mathbf{B}}}{\partial t} = 0$$ Faraday's Law

$$\vec{\nabla} \cdot \vec{\mathbf{B}} = 0$$ no magnetic monopoles

$$\vec{\nabla} \times \vec{\mathbf{B}} - \mu_0 \epsilon_0 \frac{\partial \vec{\mathbf{E}}}{\partial t} = \mu_0 \vec{\mathbf{J}}$$ Ampere-Maxwell Law

The Maxwell Equations, Cartesian components

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \frac{1}{\epsilon_0}\rho$$ Gauss's Law

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} + \frac{\partial B_x}{\partial t} = 0$$ Faraday's Law

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} + \frac{\partial B_y}{\partial t} = 0$$ Faraday's Law

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} + \frac{\partial B_z}{\partial t} = 0$$ Faraday's Law

$$\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0$$ no magnetic monopoles

$$\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} - \mu_0\epsilon_0\frac{\partial E_x}{\partial t} = \mu_0 J_x$$ Ampere-Maxwell Law

$$\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} - \mu_0\epsilon_0\frac{\partial E_y}{\partial t} = \mu_0 J_y$$ Ampere-Maxwell Law

$$\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} - \mu_0\epsilon_0\frac{\partial E_z}{\partial t} = \mu_0 J_z$$ Ampere-Maxwell Law

In the static Limit, the Maxwell equations decouple into electric equations and magnetic equations.

- Electric equations

$$\vec{\nabla} \cdot \vec{\mathbf{E}} = \frac{1}{\epsilon_0}\rho \qquad \text{Gauss's Law}$$

$$\vec{\nabla} \times \vec{\mathbf{E}} = 0 \qquad \text{Kirchhoff's Law}$$

- Magnetic equations

$$\vec{\nabla} \cdot \vec{\mathbf{B}} = 0 \qquad \text{no magnetic monopoles}$$

$$\vec{\nabla} \times \vec{\mathbf{B}} = \mu_0 \vec{\mathbf{J}} \qquad \text{Ampere's Law}$$

Integral Form of Faraday's Law from Differential Form

$$\vec{\nabla} \times \vec{\mathbf{E}} + \frac{\partial \vec{\mathbf{B}}}{\partial t} = 0$$

Choose any surface S.

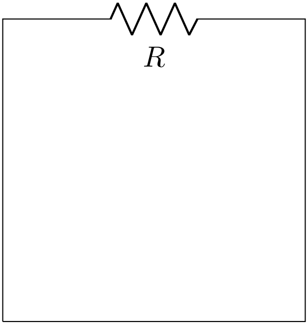
$$\int_S (\vec{\nabla} \times \vec{\mathbf{E}}) \cdot d\vec{\mathbf{a}} + \int_S \frac{\partial \vec{\mathbf{B}}}{\partial t} \cdot d\vec{\mathbf{a}} = 0$$

$$\int_S (\vec{\nabla} \times \vec{\mathbf{E}}) \cdot d\vec{\mathbf{a}} = -\frac{d}{dt} \int_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{a}}$$

$$\int_{\partial S} \vec{\mathbf{E}} \cdot d\vec{\ell} = -\frac{d}{dt} \int_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{a}}$$

$$\mathcal{E} = -\frac{d\Phi_B}{dt}$$

Beyond circuit theory



- Usually we don't care about the size of a circuit.
- Suppose we make a square of wire with side length L, and include one resistor.

Uniform magnetic field, changing in time

$$B = \alpha t$$

- Magnetic flux enclosed by the square circuit is

$$\Phi_B = \int_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{a}} = BL^2 = \alpha L^2 t.$$

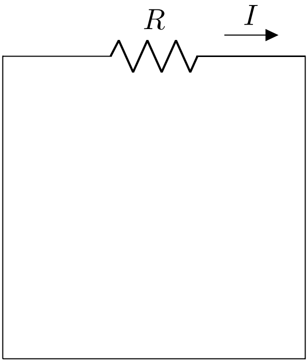
- By Faraday's law, an emf around the circuit is created.

$$\mathcal{E} = -\frac{d\Phi_B}{dt} = -\alpha L^2$$

- Negative sign means clockwise.
- A clockwise current will flow in the circuit.

$$I = \frac{\mathcal{E}}{R} = -\frac{\alpha L^2}{R}$$

Current with no battery!



$$I = \frac{\alpha L^2}{R}$$

- Kirchhoff's Voltage Law no longer holds.

The Maxwell Equations, Flux Form, Part 1

Let

- V be any volume,
- S be the boundary surface of V,
- Q be the charge contained in V,
- Φ_E be the electric flux through S, and
- Φ_B be the magnetic flux through S.

Then

$$\Phi_E = \frac{1}{\epsilon_0} Q \qquad \text{Gauss's Law}$$

$$\Phi_B = 0 \qquad \text{no magnetic monopoles}$$

The Maxwell Equations, Flux Form, Part 2

Let

- S be any surface,
- C be the boundary curve of S,
- Φ_E be the electric flux through S,
- Φ_B be the magnetic flux through S,
- C_E be the electric circulation around C,
- C_B be the magnetic circulation around C, and
- I the current through S.

Then

$$\frac{d\Phi_B}{dt} = -C_E \qquad \text{Faraday's Law}$$

$$\frac{d\Phi_E}{dt} = c^2 C_B - \mu_0 c^2 I \qquad \text{Ampere-Maxwell Law}$$

Questions

Can V be any 3-volume in 4D space-time?

The Maxwell Equations, Integral Form, Part 1

Let

- S be any surface,
- C be the boundary curve of S,
- Φ_E be the electric flux through S,
- Φ_B be the magnetic flux through S,
- C_E be the electric circulation around C,
- C_B be the magnetic circulation around C, and
- I the current through S.

Then

$$C_E + \frac{d\Phi_B}{dt} = 0 \qquad \text{Faraday's Law}$$

$$C_B - \mu_0\epsilon_0\frac{d\Phi_E}{dt} = \mu_0 I \qquad \text{Ampere-Maxwell Law}$$

The Maxwell Equations, Integral Form, Part 2

Let

- S be any *closed* surface,
- Φ_E be the electric flux through S,
- Φ_B be the magnetic flux through S,
- Q the charge enclosed by S.

Then

$$\Phi_E = \frac{1}{\epsilon_0} Q \qquad \text{Gauss's Law}$$

$$\Phi_B = 0 \qquad \text{no magnetic monopoles}$$