Electrostatics

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Static Maxwell equations

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon_0}\rho \qquad \vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = 0 \qquad \vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

- Left two equations describe electrostatics.
- Right two equations describe magnetostatics.
- In the static case, $\vec{E}$ and $\vec{B}$ do not appear together in any Maxwell equation.
- In the static case, electricity and magnetism are separate subjects.

Electrostatic Maxwell equations

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon_0} \rho$$
$$\vec{\nabla} \times \vec{E} = 0$$

- Top equation is Gauss's law.
- Bottom equation is Kirchhoff's law, the static version of Faraday's law.

Coulomb's law is the solution to Gauss's law and Kirchhoff's law.

If

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon_0}\rho \qquad\qquad \vec{\nabla} \times \vec{E} = 0$$

then

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_V \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} \rho(\vec{r}')\ dv'.$$

Electric field produced by a point particle

- Point particle has charge q, located at $\vec{r'}$.
- Field point is $\vec{r}$, where we want to know the electric field.

$$\vec{E}(\vec{r}) = \frac{q}{4\pi\epsilon_0}\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} = \frac{q}{4\pi\epsilon_0}\frac{\vec{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^3} = \frac{q}{4\pi\epsilon_0}\frac{\vec{r}-\vec{r'}}{\left|\vec{r}-\vec{r'}\right|^3}$$

Electric field produced by a point particle at the origin

$$\vec{E}(\vec{r}) = \frac{q}{4\pi\epsilon_0}\frac{\vec{r}-\vec{r'}}{\left|\vec{r}-\vec{r'}\right|^3} = \frac{q}{4\pi\epsilon_0}\frac{\vec{r}}{r^3} = \frac{q}{4\pi\epsilon_0}\frac{\hat{r}}{r^2}$$

Electric field produced by a point particle at the source point $x'\hat{x} + y'\hat{y} + z'\hat{z}$

$$\vec{E}(\vec{r}) = \frac{q}{4\pi\epsilon_0} \frac{\vec{r} - \vec{r'}}{\left|\vec{r} - \vec{r'}\right|^3}$$

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$$

$$\vec{r'} = x'\hat{x} + y'\hat{y} + z'\hat{z}$$

$$\vec{r} - \vec{r'} = (x - x')\hat{x} + (y - y')\hat{y} + (z - z')\hat{z}$$

$$\left|\vec{r} - \vec{r'}\right| = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}$$

$$\vec{E}(\vec{r}) = \frac{q}{4\pi\epsilon_0} \frac{(x - x')\hat{x} + (y - y')\hat{y} + (z - z')\hat{z}}{\left[(x - x')^2 + (y - y')^2 + (z - z')^2\right]^{3/2}}$$

Electric field produced by multiple particles

- Point particle i has charge q_i, located at $\vec{r_i'}$.
- Field point is $\vec{r}$, where we want to know the electric field.

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{\mathscr{r}_i^2}\hat{\boldsymbol{\mathscr{r}}}_i = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{\mathscr{r}_i^3}\vec{\boldsymbol{\mathscr{r}}}_i = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i(\vec{r}-\vec{r_i'})}{\left|\vec{r}-\vec{r_i'}\right|^3}$$

A little bit of charge produces a little bit of field.

- A little bit of charge dq is located at $\vec{r'}$.
- Field point is $\vec{r}$, where we want to know the electric field.

$$d\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\,dq = \frac{1}{4\pi\epsilon_0}\frac{\vec{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^3}\,dq = \frac{1}{4\pi\epsilon_0}\frac{\vec{r}-\vec{r'}}{\left|\vec{r}-\vec{r'}\right|^3}\,dq$$

$$\vec{E}(\vec{r}) = \int d\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\,dq$$

Charge Distributions

Charge distribution	Dimensionality	Symbol	SI unit
Point charge	0	q, Q	C
Linear charge density	1	λ	C/m
Surface charge density	2	σ	C/m^2
Volume charge density	3	ρ	C/m^3

Electric field produced by a line charge

- Charge is distributed along a curve C. The linear charge density at point $\vec{r'}$ is $\lambda(\vec{r'})$.
- A little bit of charge located at $\vec{r'}$ is $dq = \lambda(\vec{r'})\, d\ell'$.
- Field point is $\vec{r}$, where we want to know the electric field.

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_C \frac{\lambda(\vec{r'})}{\mathscr{r}^2}\hat{\mathscr{r}}\, d\ell' = \frac{1}{4\pi\epsilon_0}\int_C \frac{\lambda(\vec{r'})}{\mathscr{r}^3}\vec{\mathscr{r}}\, d\ell'$$
$$= \frac{1}{4\pi\epsilon_0}\int_C \frac{\lambda(\vec{r'})(\vec{r}-\vec{r'})}{\left|\vec{r}-\vec{r'}\right|^3}\, d\ell'$$

- This is a vector line integral.

Electric field produced by a surface charge

- Charge is distributed across a surface S. The surface charge density at point $\vec{r'}$ is $\sigma(\vec{r'})$.
- A little bit of charge located at $\vec{r'}$ is $dq = \sigma(\vec{r'})\, da'$.
- Field point is $\vec{r}$, where we want to know the electric field.

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_S \frac{\sigma(\vec{r'})}{\mathscr{r}^2} \hat{\boldsymbol{\mathscr{r}}}\, da' = \frac{1}{4\pi\epsilon_0} \int_S \frac{\sigma(\vec{r'})}{\mathscr{r}^3} \vec{\boldsymbol{\mathscr{r}}}\, da'$$

$$= \frac{1}{4\pi\epsilon_0} \int_S \frac{\sigma(\vec{r'})(\vec{r} - \vec{r'})}{\left|\vec{r} - \vec{r'}\right|^3}\, da'$$

- This is a vector surface integral.

Electric field produced by a volume charge

- Charge is distributed throughout a volume V. The volume charge density at point $\vec{r'}$ is $\rho(\vec{r'})$.
- A little bit of charge located at $\vec{r'}$ is $dq = \rho(\vec{r'})\, d\tau'$.
- Field point is $\vec{r}$, where we want to know the electric field.

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(\vec{r'})}{\mathscr{r}^2} \hat{\mathscr{r}}\, d\tau' = \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(\vec{r'})}{\mathscr{r}^3} \vec{\mathscr{r}}\, d\tau'$$

$$= \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(\vec{r'})(\vec{r} - \vec{r'})}{\left|\vec{r} - \vec{r'}\right|^3}\, d\tau'$$

- This is a vector volume integral.

Integrals needed to calculate electric field

notation	inputs		output
$\int_C f\, d\ell$	scalar field	curve	scalar
$\int_C \vec{F}\, d\ell$	vector field	curve	vector
$\int_C \vec{F} \cdot d\vec{\ell}$	vector field	curve	scalar
$\int_S f\, da$	scalar field	surface	scalar
$\int_S \vec{F}\, da$	vector field	surface	vector
$\int_S \vec{F} \cdot d\vec{a}$	vector field	surface	scalar
$\int_V f\, dv$	scalar field	volume	scalar
$\int_V \vec{F}\, dv$	vector field	volume	vector

- Integrals in red are the ones required to calculate electric field.