Electric Potential

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Theorem

Let $\vec{F}$ be a vector field. If

$$\vec{\nabla} \times \vec{F} = 0$$

then there exists a scalar field f such that

$$\vec{F} = \vec{\nabla} f.$$

- A conservative vector field is always the gradient of some scalar field.
- (Most vector fields are not the gradient of any scalar field.)

Electric Potential

Since the electric field is conservative

$$\vec{\nabla} \times \vec{E} = 0$$

there must be a scalar field V such that

$$\vec{E} = -\vec{\nabla} V.$$

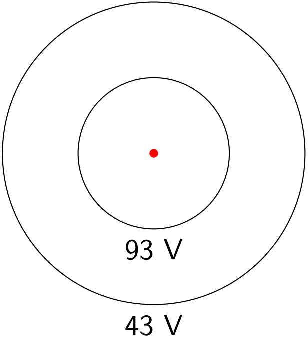
We call the scalar field V the *electric potential*.

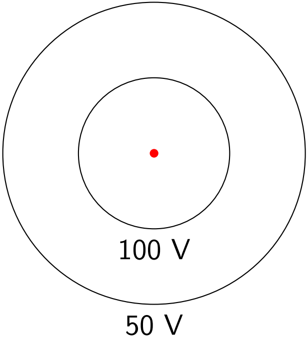
- Note the minus sign in the definition of electric potential.
- Electric field points in the direction of decreasing electric potential.

The electric field does not uniquely determine an electric potential.

- Suppose c is a constant, like 7 volts. The electric potential V and the electric potential $V + c$ give the same electric field.

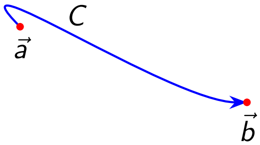
$$\vec{E} = -\vec{\nabla}(V + c) = -\vec{\nabla} V = \vec{E}$$





A second relationship between electric field and electric potential

Recall the fundamental theorem for gradients. Let C be a curve from $\vec{a}$ to $\vec{b}$.



$$\int_C \vec{\nabla} f \cdot d\vec{\ell} = f(\vec{b}) - f(\vec{a})$$

Substituting $-V$ for f and rearranging, we get

$$V(\vec{b}) - V(\vec{a}) = -\int_{\vec{a}}^{\vec{b}} \vec{E} \cdot d\vec{\ell}.$$

Finding electric potential from electric field

$$V(\vec{b}) - V(\vec{a}) = -\int_{\vec{a}}^{\vec{b}} \vec{E} \cdot d\vec{\ell}$$

If we let $\vec{a}$ be a place where electric potential is zero, then

$$V(\vec{r}) = -\int_{\vec{a}}^{\vec{r}} \vec{E} \cdot d\vec{\ell}.$$

Electric potential produced by a point particle

- Point particle has charge q, located at $\vec{r'}$.
- Field point is $\vec{r}$, where we want to know the electric field and the electric potential.

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{\mathscr{r}} = \frac{1}{4\pi\epsilon_0}\frac{q}{\left|\vec{r} - \vec{r'}\right|}$$

Electric potential produced by multiple particles

- Point particle i has charge q_i, located at $\vec{r}_i'$.
- Field point is $\vec{r}$, where we want to know the electric field and the electric potential.

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{\mathscr{r}_i} = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{\left|\vec{r} - \vec{r}_i'\right|}$$

A little bit of charge produces a little bit of electric potential.

- A little bit of charge dq is located at $\vec{r'}$.
- Field point is $\vec{r}$, where we want to know the electric field and the electric potential.

$$dV(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{dq}{\mathcal{r}} = \frac{1}{4\pi\epsilon_0}\frac{dq}{\left|\vec{r} - \vec{r'}\right|}$$

$$V(\vec{r}) = \int dV(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{dq}{\mathcal{r}}$$

Electric potential produced by a line charge

- Charge is distributed along a curve C. The linear charge density at point $\vec{r'}$ is $\lambda(\vec{r'})$.
- A little bit of charge located at $\vec{r'}$ is $dq = \lambda(\vec{r'})\, d\ell'$.
- Field point is $\vec{r}$, where we want to know the electric field.

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_C \frac{\lambda(\vec{r'})}{\scriptstyle{\mathcal{R}}}\, d\ell' = \frac{1}{4\pi\epsilon_0}\int_C \frac{\lambda(\vec{r'})}{\left|\vec{r}-\vec{r'}\right|}\, d\ell'$$

- This is a scalar line integral.

Electric potential produced by a surface charge

- Charge is distributed across a surface S. The surface charge density at point $\vec{r'}$ is $\sigma(\vec{r'})$.
- A little bit of charge located at $\vec{r'}$ is $dq = \sigma(\vec{r'})\, da'$.
- Field point is $\vec{r}$, where we want to know the electric field.

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_S \frac{\sigma(\vec{r'})}{\mathscr{r}}\, da' = \frac{1}{4\pi\epsilon_0}\int_S \frac{\sigma(\vec{r'})}{\left|\vec{r} - \vec{r'}\right|}\, da'$$

- This is a scalar surface integral.

Electric potential produced by a volume charge

- Charge is distributed throughout a volume V. The volume charge density at point $\vec{r'}$ is $\rho(\vec{r'})$.
- A little bit of charge located at $\vec{r'}$ is $dq = \rho(\vec{r'})\, d\tau'$.
- Field point is $\vec{r}$, where we want to know the electric field.

$$
\begin{aligned}
V(\vec{r}) &= \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(\vec{r'})}{\mathscr{r}}\, d\tau' \\
&= \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(\vec{r'})}{\left|\vec{r} - \vec{r'}\right|}\, d\tau'
\end{aligned}
$$

- This is a scalar volume integral.

Integrals needed to calculate electric potential from charge

notation	inputs		output	In 1.3.1?
$\int_C f \, d\ell$	scalar field	curve	scalar	no
$\int_C \vec{F} \, d\ell$	vector field	curve	vector	no
$\int_C \vec{F} \cdot d\vec{\ell}$	vector field	curve	scalar	yes
$\int_S f \, da$	scalar field	surface	scalar	no
$\int_S \vec{F} \, da$	vector field	surface	vector	no
$\int_S \vec{F} \cdot d\vec{a}$	vector field	surface	scalar	yes
$\int_V f \, dv$	scalar field	volume	scalar	yes
$\int_V \vec{F} \, dv$	vector field	volume	vector	yes