Differential Vector Calculus

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Spatial Derivatives

- In mechanics, time is *the* independent variable.
- In electromagnetic theory, there are four independent variables (x, y, z, t).
- A *field* is a function of space or spacetime.
- Electromagnetic theory is a field theory.

Spatial Derivatives

- We can take partial derivatives in each coordinate.

$$\frac{\partial}{\partial x} \qquad \frac{\partial}{\partial y} \qquad \frac{\partial}{\partial z} \qquad \frac{\partial}{\partial t}$$

The Del Operator

$$\vec{\nabla} = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$$

The Gradient

- The gradient takes a scalar field as input and produces a vector field as output.

```
grad s :: ScalarField -> VectorField
```

$$
\begin{aligned}
\vec{\nabla} f &= \left(\hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z} \right) f \\
&= \hat{x} \frac{\partial f}{\partial x} + \hat{y} \frac{\partial f}{\partial y} + \hat{z} \frac{\partial f}{\partial z} \\
&= \frac{\partial f}{\partial x} \hat{x} + \frac{\partial f}{\partial y} \hat{y} + \frac{\partial f}{\partial z} \hat{z}
\end{aligned}
$$

- The `s :: R` is a step size for the numerical derivative.

The Divergence

- The divergence takes a vector field as input and produces a scalar field as output.

```
div s :: VectorField -> ScalarField
```

$$\begin{aligned}\vec{\nabla} \cdot \vec{F} &= \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z} \right) \cdot (F_x\hat{x} + F_y\hat{y} + F_z\hat{z}) \\ &= (\hat{x}\cdot\hat{x})\frac{\partial F_x}{\partial x} + (\hat{x}\cdot\hat{y})\frac{\partial F_y}{\partial x} + \cdots \\ &= \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\end{aligned}$$

The Curl

- The curl takes a vector field as input and produces a vector field as output.

```
curl s :: VectorField -> VectorField
```

$$
\begin{aligned}
\vec{\nabla} \times \vec{F} &= \left(\hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z} \right) \times \left(F_x \hat{x} + F_y \hat{y} + F_z \hat{z} \right) \\
&= (\hat{x} \times \hat{x}) \frac{\partial F_x}{\partial x} + (\hat{x} \times \hat{y}) \frac{\partial F_y}{\partial x} + (\hat{x} \times \hat{z}) \frac{\partial F_z}{\partial x} \\
&\quad + (\hat{y} \times \hat{x}) \frac{\partial F_x}{\partial y} + (\hat{y} \times \hat{y}) \frac{\partial F_y}{\partial y} + (\hat{y} \times \hat{z}) \frac{\partial F_z}{\partial y} \\
&\quad + (\hat{z} \times \hat{x}) \frac{\partial F_x}{\partial z} + (\hat{z} \times \hat{y}) \frac{\partial F_y}{\partial z} + (\hat{z} \times \hat{z}) \frac{\partial F_z}{\partial z} \\
&= \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) \hat{x} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) \hat{y} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \hat{z}
\end{aligned}
$$

The Del Operator is a little trickier in cylindrical and spherical coordinates.

$$\vec{\nabla} = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$$

But,

$$\vec{\nabla} \neq \hat{s}\frac{\partial}{\partial s} + \hat{\phi}\frac{\partial}{\partial \phi} + \hat{z}\frac{\partial}{\partial z}$$

$$\vec{\nabla} \neq \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{\partial}{\partial \theta} + \hat{\phi}\frac{\partial}{\partial \phi}$$

Dimensions of coordinates

Coordinate	Dimensions
x	length
y	length
z	length
s	length
ϕ	angle
r	length
θ	angle

Expression	Dimensions
$\hat{x}$	dimensionless
$\hat{\phi}$	dimensionless
$\frac{\partial}{\partial x}$	inverse length
$\frac{\partial}{\partial \phi}$	dimensionless
$\vec{\nabla}$	inverse length
$\frac{1}{s}\frac{\partial}{\partial \phi}$	inverse length

Derivation of Del in Cylindrical Coordinates, page 1

$$\frac{\partial}{\partial s} = \frac{\partial x}{\partial s}\frac{\partial}{\partial x} + \frac{\partial y}{\partial s}\frac{\partial}{\partial y} + \frac{\partial z}{\partial s}\frac{\partial}{\partial z} = \cos\phi\frac{\partial}{\partial x} + \sin\phi\frac{\partial}{\partial y}$$

$$\frac{\partial}{\partial \phi} = \frac{\partial x}{\partial \phi}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \phi}\frac{\partial}{\partial y} + \frac{\partial z}{\partial \phi}\frac{\partial}{\partial z} = -s\sin\phi\frac{\partial}{\partial x} + s\cos\phi\frac{\partial}{\partial y}$$

$$\frac{\partial}{\partial z} = \frac{\partial x}{\partial z}\frac{\partial}{\partial x} + \frac{\partial y}{\partial z}\frac{\partial}{\partial y} + \frac{\partial z}{\partial z}\frac{\partial}{\partial z} = \frac{\partial}{\partial z}$$

Inverting,

$$\frac{\partial}{\partial x} = \cos\phi\frac{\partial}{\partial s} - \frac{1}{s}\sin\phi\frac{\partial}{\partial \phi}$$

$$\frac{\partial}{\partial y} = \sin\phi\frac{\partial}{\partial s} + \frac{1}{s}\cos\phi\frac{\partial}{\partial \phi}$$

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial z}$$

Derivation of Del in Cylindrical Coordinates, page 2

$$\hat{s} = \cos\phi \hat{x} + \sin\phi \hat{y}$$
$$\hat{\phi} = -\sin\phi \hat{x} + \cos\phi \hat{y}$$
$$\hat{z} = \hat{z}$$

Inverting,

$$\hat{x} = \cos\phi \hat{s} - \sin\phi \hat{\phi}$$
$$\hat{y} = \sin\phi \hat{s} + \cos\phi \hat{\phi}$$
$$\hat{z} = \hat{z}$$

Derivation of Del in Cylindrical Coordinates, page 3

$$
\begin{aligned}
\vec{\nabla} &= \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z} \\
&= \left(\cos\phi\,\hat{s} - \sin\phi\,\hat{\phi}\right)\left(\cos\phi\frac{\partial}{\partial s} - \frac{1}{s}\sin\phi\frac{\partial}{\partial \phi}\right) \\
&\quad + \left(\sin\phi\,\hat{s} + \cos\phi\,\hat{\phi}\right)\left(\sin\phi\frac{\partial}{\partial s} + \frac{1}{s}\cos\phi\frac{\partial}{\partial \phi}\right) \\
&\quad + \hat{z}\frac{\partial}{\partial z} \\
&= \hat{s}\frac{\partial}{\partial s} + \hat{\phi}\frac{1}{s}\frac{\partial}{\partial \phi} + \hat{z}\frac{\partial}{\partial z}
\end{aligned}
$$

The Del Operator in Cylindrical Coordinates

$$\vec{\nabla} \neq \hat{s}\frac{\partial}{\partial s} + \hat{\phi}\frac{\partial}{\partial \phi} + \hat{z}\frac{\partial}{\partial z}$$

$$\vec{\nabla} = \hat{s}\frac{\partial}{\partial s} + \hat{\phi}\frac{1}{s}\frac{\partial}{\partial \phi} + \hat{z}\frac{\partial}{\partial z}$$

Gradient in Cylindrical Coordinates

$$\begin{aligned}\vec{\nabla} f &= \left(\hat{s}\frac{\partial}{\partial s} + \hat{\phi}\frac{1}{s}\frac{\partial}{\partial \phi} + \hat{z}\frac{\partial}{\partial z} \right) f \\ &= \hat{s}\frac{\partial f}{\partial s} + \hat{\phi}\frac{1}{s}\frac{\partial f}{\partial \phi} + \hat{z}\frac{\partial f}{\partial z}\end{aligned}$$

Divergence in Cylindrical Coordinates

$$
\begin{aligned}
\vec{\nabla} \cdot \vec{v} &= \left(\hat{s} \frac{\partial}{\partial s} + \hat{\phi} \frac{1}{s} \frac{\partial}{\partial \phi} + \hat{z} \frac{\partial}{\partial z} \right) \cdot \left(v_s \hat{s} + v_\phi \hat{\phi} + v_z \hat{z} \right) \\
&= \hat{s} \cdot \frac{\partial}{\partial s}(v_s \hat{s}) + \hat{s} \cdot \frac{\partial}{\partial s}(v_\phi \hat{\phi}) + \hat{s} \cdot \frac{\partial}{\partial s}(v_z \hat{z}) \\
&\quad + \hat{\phi} \cdot \frac{1}{s} \frac{\partial}{\partial \phi}(v_s \hat{s}) + \hat{\phi} \cdot \frac{1}{s} \frac{\partial}{\partial \phi}(v_\phi \hat{\phi}) + \hat{\phi} \cdot \frac{1}{s} \frac{\partial}{\partial \phi}(v_z \hat{z}) \\
&\quad + \hat{z} \cdot \frac{\partial}{\partial z}(v_s \hat{s}) + \hat{z} \cdot \frac{\partial}{\partial z}(v_\phi \hat{\phi}) + \hat{z} \cdot \frac{\partial}{\partial z}(v_z \hat{z}) \\
&= \frac{\partial v_s}{\partial s} + \frac{v_s}{s} \hat{\phi} \cdot \frac{\partial \hat{s}}{\partial \phi} + \frac{1}{s} \frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z} \\
&= \frac{\partial v_s}{\partial s} + \frac{v_s}{s} + \frac{1}{s} \frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z} \\
&= \frac{1}{s} \frac{\partial}{\partial s}(s v_s) + \frac{1}{s} \frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}
\end{aligned}
$$

Curl is only a 3 by 3 determinant in Cartesian coordinates.

Determinant works in Cartesian coordinates.

$$\vec{\nabla} \times \vec{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix}$$

Determinant does not work in cylindrical coordinates.

$$\vec{\nabla} \times \vec{v} \neq \begin{vmatrix} \hat{s} & \hat{\phi} & \hat{z} \\ \frac{\partial}{\partial s} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ v_s & v_\phi & v_z \end{vmatrix}$$

Product rules

- $\vec{\nabla}(fg) = f(\vec{\nabla}g) + g(\vec{\nabla}f)$
- $\vec{\nabla} \cdot (f\vec{A}) = f(\vec{\nabla} \cdot \vec{A}) + \vec{A} \cdot (\vec{\nabla}f)$
- $\vec{\nabla} \times (f\vec{A}) = f(\vec{\nabla} \times \vec{A}) - \vec{A} \times (\vec{\nabla}f)$
- $\vec{\nabla}(\vec{A} \cdot \vec{B}) = \vec{A} \times (\vec{\nabla} \times \vec{B}) + \vec{B} \times (\vec{\nabla} \times \vec{A}) + (\vec{A} \cdot \vec{\nabla})\vec{B} + (\vec{B} \cdot \vec{\nabla})\vec{A}$
- $\vec{\nabla} \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\vec{\nabla} \times \vec{A}) - \vec{A} \cdot (\vec{\nabla} \times \vec{B})$
- $\vec{\nabla} \times (\vec{A} \times \vec{B}) = (\vec{B} \cdot \vec{\nabla})\vec{A} - (\vec{A} \cdot \vec{\nabla})\vec{B} + \vec{A}(\vec{\nabla} \cdot \vec{B}) - \vec{B}(\vec{\nabla} \cdot \vec{A})$

Second derivatives

```
grad s  ::  ScalarField -> VectorField
curl s  ::  VectorField -> VectorField
 div s  ::  VectorField -> ScalarField
```

`grad s`	`.`	`grad s`	makes no sense
`grad s`	`.`	`curl s`	makes no sense
`grad s`	`.`	`div s`	`:: VectorField -> VectorField`
`curl s`	`.`	`grad s`	`:: ScalarField -> VectorField`
`curl s`	`.`	`curl s`	`:: VectorField -> VectorField`
`curl s`	`.`	`div s`	makes no sense
`div s`	`.`	`grad s`	`:: ScalarField -> ScalarField`
`div s`	`.`	`curl s`	`:: VectorField -> ScalarField`
`div s`	`.`	`div s`	makes no sense

Two second derivatives are always zero.

```
grad s  .  div s   ::  VectorField -> VectorField
curl s  .  grad s  ::  ScalarField -> VectorField   zero
curl s  .  curl s  ::  VectorField -> VectorField
 div s  .  grad s  ::  ScalarField -> ScalarField
 div s  .  curl s  ::  VectorField -> ScalarField   zero
```

$$\vec{\nabla} \times (\vec{\nabla} f) = 0$$

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{F}) = 0$$

- s is the step size to use in the derivative

The divergence of the gradient is called the *Laplacian*.

```
grad s  .  div s   ::  VectorField -> VectorField
curl s  .  curl s  ::  VectorField -> VectorField
 div s  .  grad s  ::  ScalarField -> ScalarField
```

$$\nabla^2 f = \vec{\nabla} \cdot (\vec{\nabla} f)$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$