Coordinate Systems

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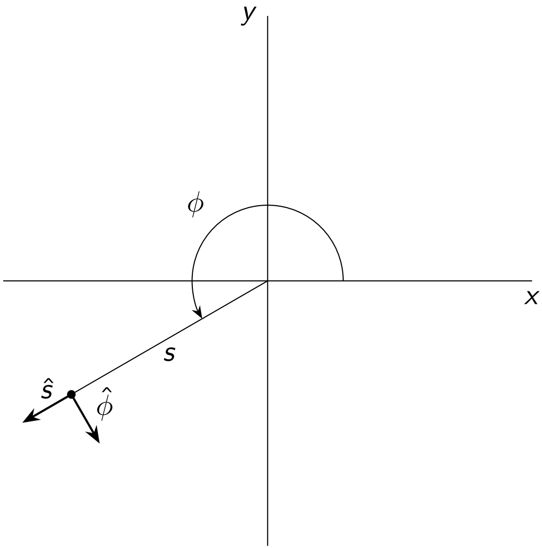
September 22, 2021

Polar Coordinates



Relationship between Polar Coordinates and 2D Cartesian Coordinates

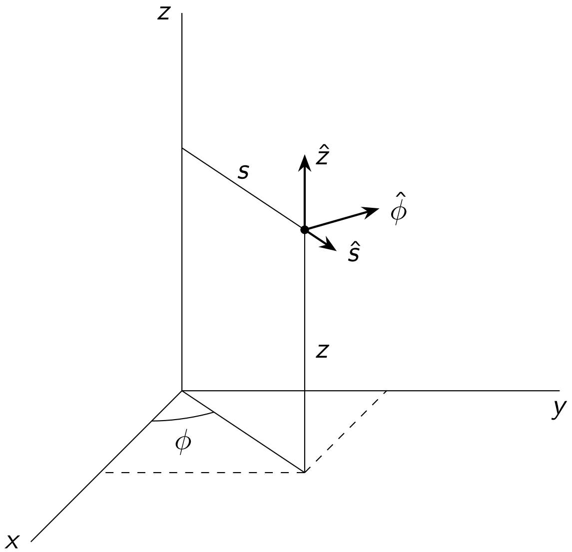
$$x = s\cos\phi \qquad\qquad s = \sqrt{x^2 + y^2}$$

$$y = s\sin\phi \qquad\qquad \phi = \tan^{-1}\frac{y}{x}$$

$$\hat{s} = \frac{x\hat{x} + y\hat{y}}{\sqrt{x^2 + y^2}} = \cos\phi\hat{x} + \sin\phi\hat{y}$$

$$\hat{\phi} = -\sin\phi\hat{x} + \cos\phi\hat{y}$$

Cylindrical Coordinates

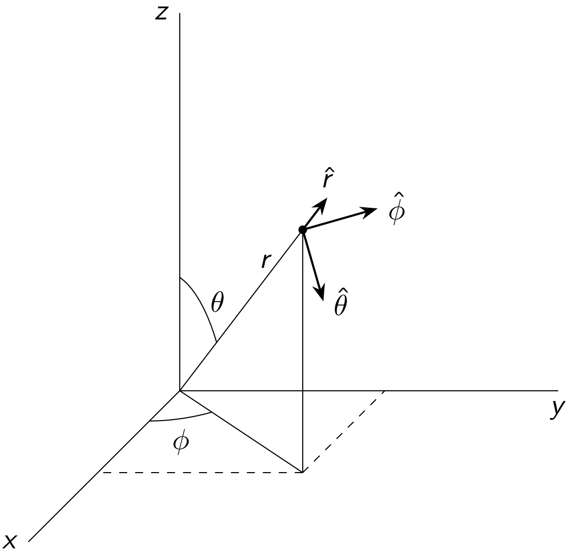


Relationship between Cylindrical Coordinates and 3D Cartesian Coordinates

$$x = s\cos\phi$$
$$y = s\sin\phi$$
$$z = z$$

$$\hat{s} = \cos\phi\hat{x} + \sin\phi\hat{y}$$
$$\hat{\phi} = -\sin\phi\hat{x} + \cos\phi\hat{y}$$
$$\hat{z} = \hat{z}$$

Spherical Coordinates



Relationship between Spherical Coordinates and 3D Cartesian Coordinates

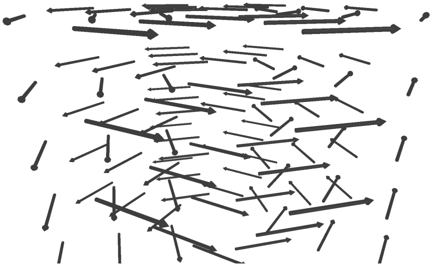
$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

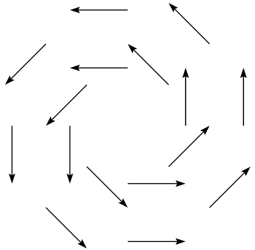
$$\hat{r} = \frac{x\hat{x} + y\hat{y} + z\hat{z}}{\sqrt{x^2 + y^2 + z^2}} = \sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi\hat{y} + \cos\theta\hat{z}$$
$$\hat{\theta} = \cos\theta\cos\phi\hat{x} + \cos\theta\sin\phi\hat{y} - \sin\theta\hat{z}$$
$$\hat{\phi} = -\sin\phi\hat{x} + \cos\phi\hat{y}$$

Some unit vectors are not uniform.

- $\hat{x}$, $\hat{y}$, $\hat{z}$, $\hat{s}$, $\hat{\phi}$, $\hat{r}$, and $\hat{\theta}$ are really unit vector *fields*, because their direction depends on location in space.
- $\hat{x}$, $\hat{y}$, and $\hat{z}$ are uniform. Each points in the same direction, regardless of position in space.
- $\hat{s}$, $\hat{\phi}$, $\hat{r}$, and $\hat{\theta}$ are not uniform.
- You can bring $\hat{x}$, $\hat{y}$, and $\hat{z}$ outside of an integral, because they are constant (uniform).
- You cannot bring $\hat{s}$, $\hat{\phi}$, $\hat{r}$, or $\hat{\theta}$ outside of a spatial integral without checking whether the unit vector depends on the variable(s) of integration.

Two views of the vector field $\hat{\phi}$





Dependence of unit vectors on coordinates.

Unit vector	depends on
$\hat{x}$	nothing
$\hat{y}$	nothing
$\hat{z}$	nothing
$\hat{s}$	ϕ
$\hat{\phi}$	ϕ
$\hat{r}$	θ, ϕ
$\hat{\theta}$	θ, ϕ

Upgraded line integral advice

- Use the differential line element appropriate to your coordinate system.

$$d\vec{\ell} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$$
$$d\vec{\ell} = ds\,\hat{s} + s\,d\phi\,\hat{\phi} + dz\,\hat{z}$$
$$d\vec{\ell} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi}$$

- Pick *one* variable (usually x, y, z, s, ϕ, r, or θ) to be your variable of integration.
- Use knowledge of the curve to express all coordinates in terms of your one variable.
- Choose limits for your one variable of integration from knowledge of the curve.
- Once you reduce the line integral to an ordinary integral in one variable, evaluate that integral to get the result.

Upgraded surface integral advice

- Choose a differential surface element $d\vec{a}$ from knowledge of the surface. Usually, the surfaces we deal with have one coordinate that is constant across the surface. Identify that coordinate and use this table:

Constant	differential surface element		
x	$d\vec{a} = dy\, dz\, \hat{x}$	or	$d\vec{a} = -dy\, dz\, \hat{x}$
y	$d\vec{a} = dx\, dz\, \hat{y}$	or	$d\vec{a} = -dx\, dz\, \hat{y}$
z	$d\vec{a} = dx\, dy\, \hat{z}$	or	$d\vec{a} = -dx\, dy\, \hat{z}$
s	$d\vec{a} = s\, d\phi\, dz\, \hat{s}$	or	$d\vec{a} = -s\, d\phi\, dz\, \hat{s}$
ϕ	$d\vec{a} = ds\, dz\, \hat{\phi}$	or	$d\vec{a} = -ds\, dz\, \hat{\phi}$
z	$d\vec{a} = ds\, s\, d\phi\, \hat{z}$	or	$d\vec{a} = -ds\, s\, d\phi\, \hat{z}$
r	$d\vec{a} = r\, d\theta\, r \sin\theta\, d\phi\, \hat{r}$	or	$d\vec{a} = -r\, d\theta\, r \sin\theta\, d\phi\, \hat{r}$
θ	$d\vec{a} = dr\, r \sin\theta\, d\phi\, \hat{\theta}$	or	$d\vec{a} = -dr\, r \sin\theta\, d\phi\, \hat{\theta}$
ϕ	$d\vec{a} = dr\, r\, d\theta\, \hat{\phi}$	or	$d\vec{a} = -dr\, r\, d\theta\, \hat{\phi}$

Upgraded surface integral advice, continued

- If your surface has a coordinate with a constant value across the surface, plug in that constant value.
- Choose limits for your two variables of integration from knowledge of the surface.
- Once you reduce the surface integral to an ordinary double integral in two variables, evaluate that integral to get the result.

Updated volume integral advice

- Use the differential volume element $d\tau$ appropriate to your coordinate system.

$$d\tau = dx\, dy\, dz$$
$$d\tau = s\, ds\, d\phi\, dz$$
$$d\tau = r^2 \sin\theta\, dr\, d\theta\, d\phi$$

- For the innermost (first of 3) integration, think of the lower and upper limits as surfaces. To describe such a surface, write the first (innermost) variable of integration in terms of the other two variables.
- For the second integration, think of the lower and upper limits as curves.
- For the third and final integration, think of the lower and upper limits as numbers.
- Evaluate the triple integral to get a result.