Ampere's Law

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Ampere's Law

$$\vec{\nabla} \times \vec{\mathbf{B}} = \mu_0 \vec{\mathbf{J}}$$

- The curl of magnetic field at a point in space is proportional to the current density at that point.
- Where current points in the direction of your right thumb, magnetic field points in the direction of your bent right fingers.
- The equation above is called the *differential form* of Ampere's law.

Deriving the integral form of Ampere's Law

- Start with the differential form of Ampere's law.

$$\vec{\nabla} \times \vec{\mathbf{B}} = \mu_0 \vec{\mathbf{J}}$$

- Pick any oriented surface S. Do a flux integral of both sides of Ampere's law using this surface S.

$$\int_S (\vec{\nabla} \times \vec{\mathbf{B}}) \cdot d\vec{\mathbf{a}} = \mu_0 \int_S \vec{\mathbf{J}} \cdot d\vec{\mathbf{a}}$$

- Use Stokes' theorem to rewrite the left side. Realize that the integral on the right side is the current flowing through the surface.

$$\int_{\partial S} \vec{\mathbf{B}} \cdot d\vec{\ell} = \mu_0 I_{\text{enc}}$$

- The term on the left is the magnetic circulation around the boundary of S.

Integral form of Ampere's Law

$$\int_C \vec{\mathbf{B}} \cdot d\vec{\ell} = \mu_0 I_{\text{enc}}$$

- C is any closed curve or loop.
- I_{enc} is the current flowing through C.
- The magnetic circulation around a closed loop C is proportional to the current enclosed by the loop.
- Ampere's law is not a recipe for finding anything.
- It is a claim that magnetic field and current are related.
- Current gives rise to magnetic field.
- If a current distribution has high symmetry, we can use the integral form of Ampere's law to find the magnetic field produced by the current distribution.

Using Ampere's law to find the magnetic field when the current distribution has cylindrical symmetry

- Any rotation about the symmetry axis leaves the current distribution unchanged, and hence must leave the magnetic field unchanged.
- Any translation parallel to the symmetry axis leaves the current distribution unchanged, and hence must leave the magnetic field unchanged.
- Cylindrical symmetry has two cases, one for current distributions that are like a long straight wire, and one for distributions that are like a solenoid.

Cylindrical Symmetry Case 1

- $\vec{\mathbf{J}}$ depends only on s, not on ϕ or z, and $\vec{\mathbf{J}}$ points in the $\hat{\mathbf{z}}$ direction.
$$\vec{\mathbf{J}} = J(s)\hat{\mathbf{z}}$$
- The magnetic field must point in the $\hat{\boldsymbol{\phi}}$ direction: $\vec{\mathbf{B}} = B(s)\hat{\boldsymbol{\phi}}$.
- The azimuthal component $B(s)$ of the magnetic field can depend only on s, not on ϕ or z.
- Choose a circle of radius s as your Amperian loop C. Notice that
$$\int_C \vec{\mathbf{B}} \cdot d\vec{\ell} = B(s) 2\pi s.$$

Cylindrical Symmetry Case 2

- $\vec{\mathbf{J}}$ points in the $\hat{\phi}$ direction; the magnitude of $\vec{\mathbf{J}}$ depends only on s, not on ϕ or z.
$$\vec{\mathbf{J}} = J(s)\hat{\boldsymbol{\phi}}$$
- The magnetic field must point in the $\hat{\mathbf{z}}$ direction: $\vec{\mathbf{B}} = B(s)\hat{\mathbf{z}}$.
- The $\hat{\mathbf{z}}$ component $B(s)$ of the magnetic field can depend only on s, not on ϕ or z.
- Choose a rectangle as your Amperian loop C. The sides parallel to the $\hat{\mathbf{z}}$ direction have length L. The sides parallel to the $\hat{\mathbf{s}}$ direction extend from s_1 to s_2. Notice that
$$\int_C \vec{\mathbf{B}} \cdot d\vec{\ell} = B(s_2)L - B(s_1)L.$$

Using Ampere's law to find the magnetic field when the current distribution has planar symmetry

- With planar symmetry, $\vec{\mathbf{J}}$ points in the $\hat{\mathbf{x}}$ direction; the magnitude of $\vec{\mathbf{J}}$ depends only on z, not on x or y.

$$\vec{\mathbf{J}} = J(z)\hat{\mathbf{x}}$$

- Any translation in the xy plane leaves the current distribution unchanged, and hence must leave the magnetic field unchanged.
- The magnetic field must point in the $\hat{\mathbf{y}}$ direction: $\vec{\mathbf{B}} = B(z)\hat{\mathbf{y}}$.
- The $\hat{\mathbf{y}}$ component $B(z)$ of the magnetic field can depend only on z, not on x or y.
- Choose a rectangle as your Amperian loop C. The sides parallel to the $\hat{\mathbf{y}}$ direction have length L. The sides parallel to the $\hat{\mathbf{z}}$ direction extend from z_1 to z_2. Notice that

$$\int_C \vec{\mathbf{B}} \cdot d\vec{\ell} = B(z_1)L - B(z_2)L.$$