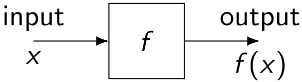
Function Composition

Scott N. Walck

September 5, 2022

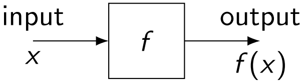
A function is a machine that produces an output from an input.



- f is the name of the function
- x is the name of the input
- $f(x)$ is the name of the output

Example: the squaring function

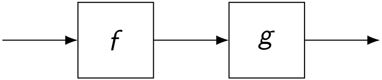
$$f(x) = x^2$$



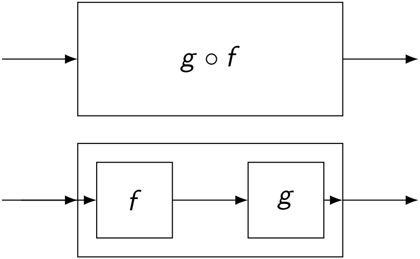
input x	output $f(x)$
-3	9
-2	4
-1	1
0	0
1	1
2	4
3	9

Composition is a particular kind of plumbing.

- Use the output of one function as the input to another.



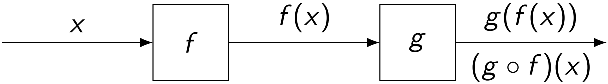
- The composite function is denoted $g \circ f$, called "g after f".



Definition of the composite function

We define the composite function $g \circ f$ by saying what it does.

$$(g \circ f)(x) = g(f(x))$$



Composition Example

$$f(x) = x + 1 \qquad g(x) = x^2$$

$$(g \circ f)(x) = g(f(x)) = g(x+1) = (x+1)^2$$
$$(f \circ g)(x) = f(g(x)) = f(x^2) = x^2 + 1$$

- Composition of functions is usually not commutative.

$$g \circ f \neq f \circ g$$

Identity function

- The identity function i return its input as its output.

$$i(x) = x$$

- Seems silly to name this function, but it turns out to be useful.

Some functions have an inverse; some don't.

- Suppose we have a function $f : D \to C$. (D is the domain and C is the codomain.)
- If there is a function $g : C \to D$ such that

  $$g \circ f = i$$

  and

  $$f \circ g = i$$

  then we say that f is *invertible* and the function g is its inverse. In this case we often write the function g as f^{-1}. (Mathematicians will want to prove that a function f can have at most one inverse. We will take this for granted.)

Inverse Examples

$$f(x) = x + 1 \qquad\qquad g(x) = x^2$$

- f has an inverse.
- g does not. (Assuming we view $g : \mathbb{R} \to \mathbb{R}$.)

Inverse Example 1

$$f(x) = x + 1$$

$$f^{-1}(x) = x - 1$$

$$(f^{-1} \circ f)(x) = f^{-1}(f(x)) = f^{-1}(x + 1) = x + 1 - 1 = x$$
$$(f \circ f^{-1})(x) = f(f^{-1}(x)) = f(x - 1) = x - 1 + 1 = x$$

Inverse Example 2

$$g(x) = x^2$$

$$g^{-1}(x) = \sqrt{x}$$

$$(g^{-1} \circ g)(-5) = g^{-1}(g(-5)) = g^{-1}(25) = 5 \neq -5$$

- $g : \mathbb{R} \to \mathbb{R}$ has no inverse, but maybe we can fix it.

An invertible squaring function

- If we restrict the domain of the squaring function to nonnegative numbers, then it is invertible.

$$g(x) = x^2$$

$$g^{-1}(x) = \sqrt{x}$$

$$(g^{-1} \circ g)(x) = g^{-1}(g(x)) = g^{-1}(x^2) = \sqrt{x^2} = |x| = x$$
$$(g \circ g^{-1})(x) = g(g^{-1}(x)) = g(\sqrt{x}) = (\sqrt{x})^2 = x$$

- Let $\mathbb{R}_{\geq 0}$ denote the nonnegative real numbers. The function $g : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is invertible.