Stern-Gerlach Experiment

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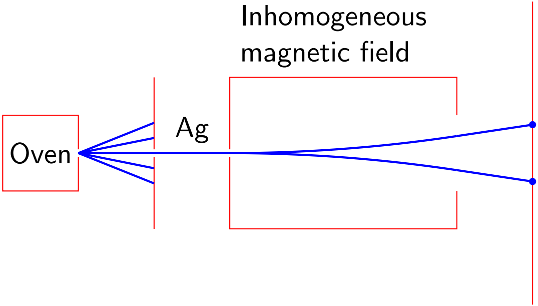
Stern-Gerlach experiment (1922)



- expect a range in amount deflection
- get exactly *two* types of deflection
- puts the *quantum* in quantum mechanics
- Ag acts like a spin-1/2 particle

Spin-1/2 particle

- Any particle that has two outcomes in a Stern-Gerlach experiment is a spin-1/2 particle
- Some spin-1/2 particles
  - The electron
  - The proton
  - The neutron
  - Every quark
- A *qubit* is any quantum system for which some two-outcome experiment can be performed (doesn't need to be a Stern-Gerlach experiment).
- Every quantity associated with a qubit has two possible values (for example, Σ_1, Σ_2, and Σ_3 are quantities associated with a qubit).

Pauli Vector

	1-direction	2-direction	3-direction
Pauli vector	Σ_1	Σ_2	Σ_3
Spin angular momentum	$S_1 = \frac{\hbar}{2}\Sigma_1$	$S_2 = \frac{\hbar}{2}\Sigma_2$	$S_3 = \frac{\hbar}{2}\Sigma_3$
Magnetic moment	$\mu\Sigma_1$	$\mu\Sigma_2$	$\mu\Sigma_3$

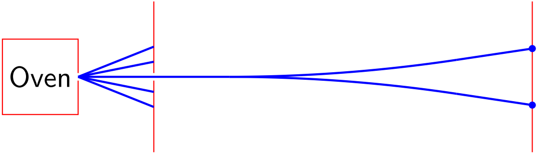
- Three vectors that contain the same information
- Magnetic moment is most readily measured
- We will work mostly with the Pauli vector

Possible values

	1-direction	2-direction	3-direction
Pauli vector	Σ_1	Σ_2	Σ_3
possible values	$-1, 1$	$-1, 1$	$-1, 1$
Spin angular momentum	$S_1 = \frac{\hbar}{2}\Sigma_1$	$S_2 = \frac{\hbar}{2}\Sigma_2$	$S_3 = \frac{\hbar}{2}\Sigma_3$
possible values	$-\frac{\hbar}{2}, \frac{\hbar}{2}$	$-\frac{\hbar}{2}, \frac{\hbar}{2}$	$-\frac{\hbar}{2}, \frac{\hbar}{2}$
Magnetic moment	$\mu\Sigma_1$	$\mu\Sigma_2$	$\mu\Sigma_3$
possible values	$-\mu, \mu$	$-\mu, \mu$	$-\mu, \mu$

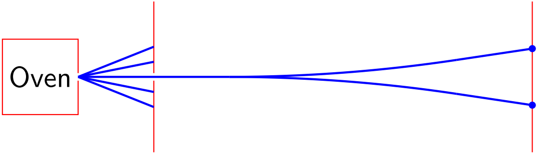
Stern-Gerlach outcomes

Measurement in the x direction (1 direction):



Σ_1	S_1	$\mu\Sigma_1$
1	$\frac{\hbar}{2}$	μ
-1	$-\frac{\hbar}{2}$	$-\mu$

Measurement in the y direction (2 direction):



Σ_2	S_2	$\mu\Sigma_2$
1	$\frac{\hbar}{2}$	μ
-1	$-\frac{\hbar}{2}$	$-\mu$

The totally mixed state of one qubit

Is it possible to have the following probabilities?

- In the 1 direction: $\rho(-1) = 1/2$, $\rho(1) = 1/2$
- In the 2 direction: $\rho(-1) = 1/2$, $\rho(1) = 1/2$
- In the 3 direction: $\rho(-1) = 1/2$, $\rho(1) = 1/2$

Yes!

Because $\langle \Sigma_1 \rangle = \langle \Sigma_2 \rangle = \langle \Sigma_3 \rangle = 0$ and so
$\langle \Sigma_1 \rangle^2 + \langle \Sigma_2 \rangle^2 + \langle \Sigma_3 \rangle^2 \leq 1$.

An impossible situation

Is it possible to have the following probabilities?

- In the 1 direction: $\rho(-1) = 0$, $\rho(1) = 1$
- In the 2 direction: $\rho(-1) = 0$, $\rho(1) = 1$
- In the 3 direction: $\rho(-1) = 1/2$, $\rho(1) = 1/2$

No!

Because $\langle \Sigma_1 \rangle = \langle \Sigma_2 \rangle = 1$ and so $\langle \Sigma_1 \rangle^2 + \langle \Sigma_2 \rangle^2 + \langle \Sigma_3 \rangle^2 \not\leq 1$.

- If a particle has a definite value in one direction, it cannot have a definite value in any other direction.

Knowing the probabilities for a Pauli matrix is equivalent to knowing the mean value of the Pauli matrix.

Every Pauli matrix has possible values -1 and 1.
Take Σ_1, for example.

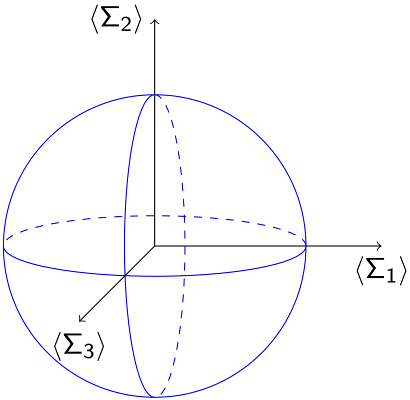
$$\langle \Sigma_1 \rangle = \rho(-1)(-1) + \rho(1)(1)$$

$$\rho(-1) = \frac{1}{2} - \frac{1}{2}\langle \Sigma_1 \rangle$$
$$\rho(1) = \frac{1}{2} + \frac{1}{2}\langle \Sigma_1 \rangle$$

Which probabilities are possible?

A collection of probability distributions for Σ_1, Σ_2, and Σ_3 is possible if and only if

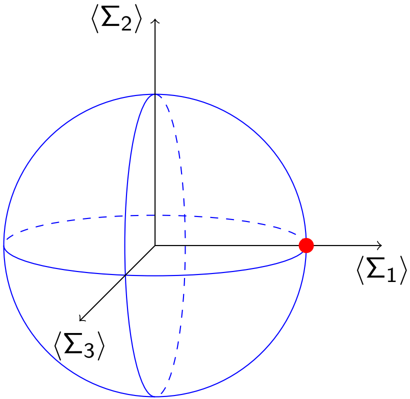
$$\langle \Sigma_1 \rangle^2 + \langle \Sigma_2 \rangle^2 + \langle \Sigma_3 \rangle^2 \leq 1.$$



- Points inside and on the sphere are possible states
- Points outside the sphere are not possible

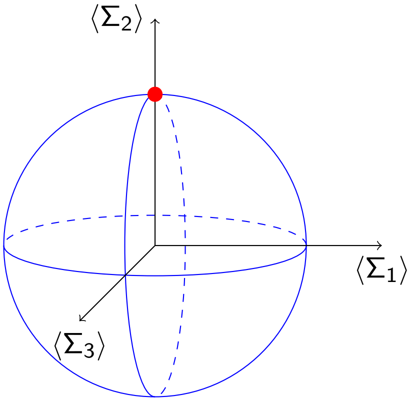
A spin-1/2 particle is a qubit

$\langle \Sigma_1 \rangle = 1$ $\langle \Sigma_2 \rangle = 0$ $\langle \Sigma_3 \rangle = 0$



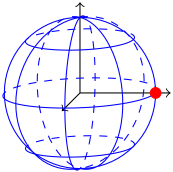
Bloch sphere describes the state of a qubit

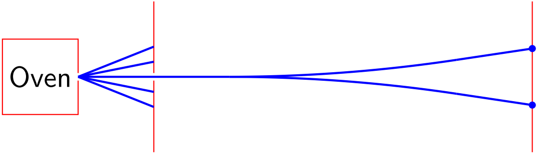
$\langle \Sigma_1 \rangle = 0$ $\langle \Sigma_2 \rangle = 1$ $\langle \Sigma_3 \rangle = 0$



Measurement in the x direction (1 direction)

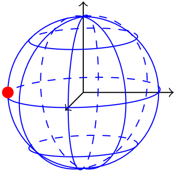
$$\langle \Sigma_1 \rangle = 1$$
$$\langle \Sigma_2 \rangle = 0$$
$$\langle \Sigma_3 \rangle = 0$$





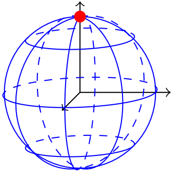
Σ_1	S_1	$\mu\Sigma_1$
1	$\frac{\hbar}{2}$	μ
-1	$-\frac{\hbar}{2}$	$-\mu$

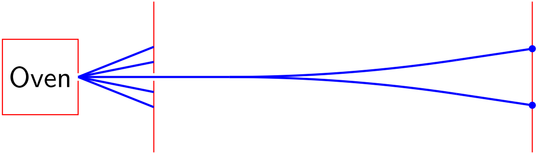
$$\langle \Sigma_1 \rangle = -1$$
$$\langle \Sigma_2 \rangle = 0$$
$$\langle \Sigma_3 \rangle = 0$$



Measurement in the y direction (2 direction)

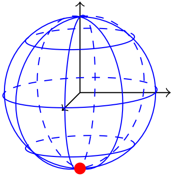
$$\langle \Sigma_1 \rangle = 0$$
$$\langle \Sigma_2 \rangle = 1$$
$$\langle \Sigma_3 \rangle = 0$$





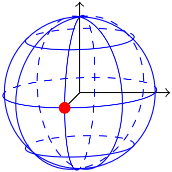
Σ_2	S_2	$\mu\Sigma_2$
1	$\frac{\hbar}{2}$	μ
-1	$-\frac{\hbar}{2}$	$-\mu$

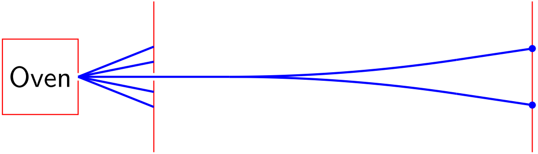
$$\langle \Sigma_1 \rangle = 0$$
$$\langle \Sigma_2 \rangle = -1$$
$$\langle \Sigma_3 \rangle = 0$$



Measurement in the z direction (3 direction)

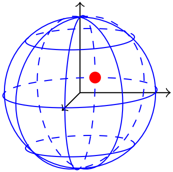
$$\langle \Sigma_1 \rangle = 0$$
$$\langle \Sigma_2 \rangle = 0$$
$$\langle \Sigma_3 \rangle = 1$$





Σ_3	S_3	$\mu\Sigma_3$
1	$\frac{\hbar}{2}$	μ
-1	$-\frac{\hbar}{2}$	$-\mu$

$$\langle \Sigma_1 \rangle = 0$$
$$\langle \Sigma_2 \rangle = 0$$
$$\langle \Sigma_3 \rangle = -1$$



The totally mixed state of one qubit

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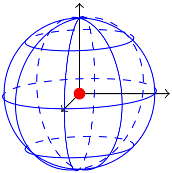
- In the 3 direction: $\rho(-1) = 1/2$, $\rho(1) = 1/2$

Yes!
Because $\langle \Sigma_1 \rangle = \langle \Sigma_2 \rangle = \langle \Sigma_3 \rangle = 0$ and so $\langle \Sigma_1 \rangle^2 + \langle \Sigma_2 \rangle^2 + \langle \Sigma_3 \rangle^2 \leq 1$.



An impossible situation

Is it possible to have the following probabilities?

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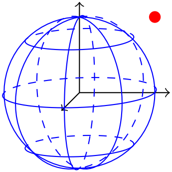
- In the 3 direction: $\rho(-1) = 1/2$, $\rho(1) = 1/2$

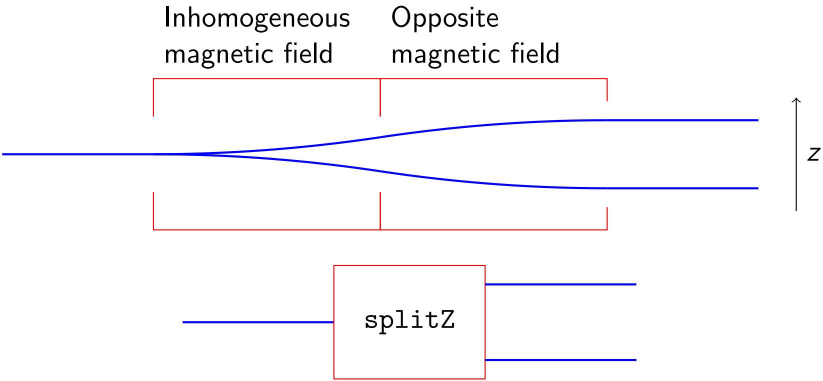
No!

Because $\langle \Sigma_1 \rangle = \langle \Sigma_2 \rangle = 1$ and so $\langle \Sigma_1 \rangle^2 + \langle \Sigma_2 \rangle^2 + \langle \Sigma_3 \rangle^2 \not\leq 1$.

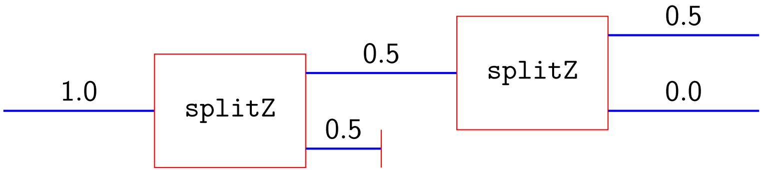
- If a particle has a definite value in one direction, it cannot have a definite value in any other direction.



Stern-Gerlach beam splitter

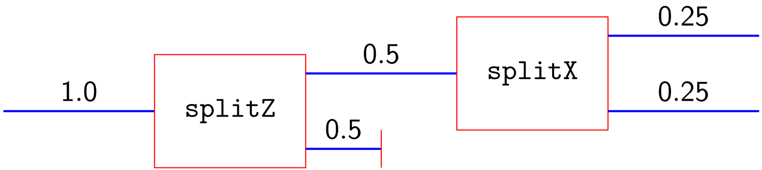


Townsend's Experiment 1: Reproducibility

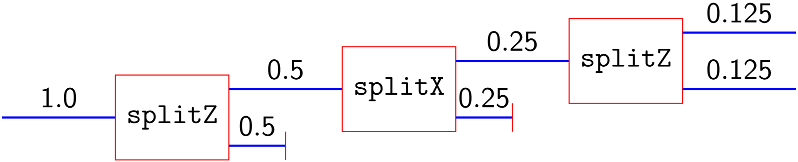


- Every particle that exits with a value of 1 will exit again with a value of 1 if measured in the same direction.
- Every particle that exits with a value of -1 will exit again with a value of -1 if measured in the same direction.
- Results are not completely random for every measurement. There is some predictability.

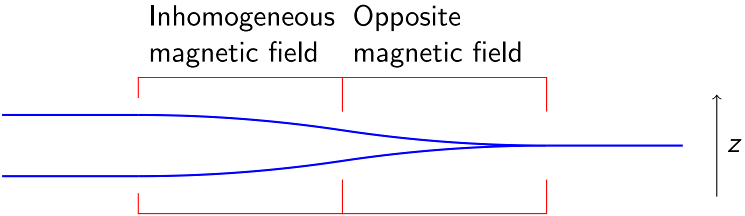
Townsend's Experiment 2: Z then X

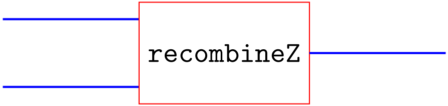


Townsend's Experiment 3: Z then X then Z

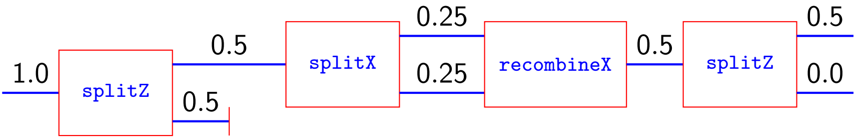


Stern-Gerlach beam recombiner





Townsend's Experiment 4: Recombination



Possible values

- The possible values of Σ_1 are -1 and 1.
- The possible values of

$$x_1\Sigma_1 + x_2\Sigma_2 + x_3\Sigma_3$$

are

$$-r, r$$

where

$$r = \sqrt{x_1^2 + x_2^2 + x_3^2}.$$

Possible values

- The possible values of

$$x_0 1 + x_1 \Sigma_1 + x_2 \Sigma_2 + x_3 \Sigma_3$$

are

$$x_0 - r, x_0 + r$$

where

$$r = \sqrt{x_1^2 + x_2^2 + x_3^2}.$$

Possible values

- The possible values of

$$M = z_0 1 + c(x_1\Sigma_1 + x_2\Sigma_2 + x_3\Sigma_3)$$

are

$$z_0 - cr, z_0 + cr$$

where

$$r = \sqrt{x_1^2 + x_2^2 + x_3^2}.$$

State of a spin-1/2 particle

How can we describe the *state* of a spin-1/2 particle?

- Method 1: Specify three real numbers $\langle \Sigma_1 \rangle$, $\langle \Sigma_2 \rangle$, and $\langle \Sigma_3 \rangle$ that satisfy
$$\langle \Sigma_1 \rangle^2 + \langle \Sigma_2 \rangle^2 + \langle \Sigma_3 \rangle^2 \leq 1.$$
- Method 2: Specify a definite value for a physical quantity. (The definite value must be one of the possible values of the physical quantity.)

Converting from Method 2 to Method 1

- The possible values of $M = x_0 1 + x_1\Sigma_1 + x_2\Sigma_2 + x_3\Sigma_3$ are

$$x_0 - r, x_0 + r$$

where

$$r = \sqrt{x_1^2 + x_2^2 + x_3^2}.$$

If M has the definite value $x_0 - r$, then

$$\langle\Sigma_1\rangle = -\frac{x_1}{r}$$
$$\langle\Sigma_2\rangle = -\frac{x_2}{r}$$
$$\langle\Sigma_3\rangle = -\frac{x_3}{r}.$$

If M has the definite value $x_0 + r$, then

$$\langle\Sigma_1\rangle = \frac{x_1}{r}$$
$$\langle\Sigma_2\rangle = \frac{x_2}{r}$$
$$\langle\Sigma_3\rangle = \frac{x_3}{r}.$$

Outcome probabilities

- The possible values of

$$M = x_0 1 + x_1\Sigma_1 + x_2\Sigma_2 + x_3\Sigma_3$$

  are

$$x_0 - r, x_0 + r$$

  where

$$r = \sqrt{x_1^2 + x_2^2 + x_3^2}.$$

- The probabilities for these outcomes are

$$\rho(x_0 - r) = \frac{1}{2} - \frac{1}{2}\langle U\rangle \qquad \rho(x_0 + r) = \frac{1}{2} + \frac{1}{2}\langle U\rangle$$

  where

$$U = \frac{x_1}{r}\Sigma_1 + \frac{x_2}{r}\Sigma_2 + \frac{x_3}{r}\Sigma_3.$$

Collapse of the wavepacket

- Typically, before a measurement, a physical quantity M does not have a definite value. There are different possibilities for the outcome of a measurement of M.
- After we measure physical quantity M and get a particular outcome m (either $x_0 - r$ or $x_0 + r$), the new state of the particle is one in which the physical quantity M has the definite value m.
- We say that the state has collapsed to one in which M has the definite value m.

Spin-1/2 summary

- What is a spin-1/2 particle?
- What kind of measurements can I make on a spin-1/2 particle?
- What are the possible outcomes I can get from these measurements?
- Can I predict in advance what outcome I will get?

Measurement postulate for a spin-1/2 particle

1. A number of real physical quantities can be measured on a spin-1/2 particle. A real physical quantity is described by a matrix

$$x_0 1 + x_1 \Sigma_1 + x_2 \Sigma_2 + x_3 \Sigma_3$$

   in which x_0, x_1, x_2, and x_3 are real numbers.

2. Each real physical quantity has two possible outcomes. We will get one of these outcomes when we make the measurement.