Orthogonal coordinates on 4 dimensional Kähler manifolds

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Abstract

C. F. Gauss constructed coordinates on any surface in space so that $F = 0$, that is, so that the coordinate directions were orthogonal in a neighborhood. In 1984, Dennis DeTurck and Dean Yang showed the existence of orthogonal coordinates on any Riemannian 3-manifold. They also showed that, for dimensions at least 4, there is a curvature obstruction to the existence of orthogonal coordinates, in that curvature components of the form R_{ijkl}, with all 4 indices distinct, will vanish if the directions correspond to orthogonal coordinates.

Recently, Paul Gauduchon and Andrei Moroianu showed that there are no orthogonal coordinates on $\mathbb{CP}^n$ or $\mathbb{HP}^n$, with the standard metrics, if $n > 1$. In the case of $\mathbb{CP}^2$, the curvature condition is inadequate to show their result; a mysterious trick is used instead.

Today's talk will focus on 4 (real)-dimensional Kähler manifolds, their elegant and special curvature, and how the underlying complex-analytic structure lies behind Gauduchon and Moroianu's result, which reveals further obstructions to the existence of orthogonal coordinates.

Curvature

Given a manifold M with Riemannian metric, $g_{ij} := \left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right\rangle$, B. Riemann constructed the *curvature tensor* to measure the shape, or curvature, of the space. Geometrically, this tensor gives the Gaussian curvature for each 2-dimensional direction plane at each point. The formulation begins with the *covariant derivative* $\nabla_X Y$, measuring how a vector field Y deviates from parallel as you move in the direction of a tangent vector X. What accomplishes that is the expression, called the *Riemann-Christoffel curvature tensor*, for each triple of vector fields X, Y, Z,

$$R_{XY}Z := \nabla_{[X,Y]}Z - \nabla_X(\nabla_Y Z) + \nabla_Y(\nabla_X Z),$$

from which you can compute the Gaussian curvature in the directions spanned by X and Y by the "diagonal" terms $\langle R_{XY}X, Y \rangle$. The idea is that this tensor gives the deviation from flatness, the geometry, of the space.

Orthogonal coordinates

A Riemannian manifold M^n with Riemannian metric $\langle\, ,\, \rangle$ has *orthogonal coordinates* $\{x_1, \ldots, x_n\}$ in a neighborhood U if, for each point $x \in U$, $\left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right\rangle = 0$ whenever $i \neq j$. If each point of M has orthogonal coordinates in some neighborhood, then we say that *M has orthogonal coordinates* or *supports orthogonal coordinates*. Given an orthogonal coordinate system, set $a_i := \left\| \frac{\partial}{\partial x_i} \right\| = \sqrt{\left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_i} \right\rangle}$. The *associated* frame $\{e_1, \ldots, e_n\}$ is defined by setting $e_i := \frac{1}{a_i} \frac{\partial}{\partial x_i}$. From the definitions of the covariant derivative and Riemann curvature tensor, Gauduchon and Moroianu show the following useful results.

Proposition

[Gauduchon and Moroianu] *Let M be a Riemannian manifold with orthogonal coordinates on a chart U, with associated frame $\{e_1, \dots, e_n\}$. Then,*

1. *If $i \neq j$, then $\nabla_{e_i} e_j = \frac{1}{a_i} e_j(a_i)\, e_i$, so $[e_i, e_j] = \frac{1}{a_i} e_j(a_i)\, e_i - \frac{1}{a_j} e_i(a_j)\, e_j$.*
2. $\nabla_{e_i} e_i = -\sum_{j \neq i} \frac{1}{a_i} e_j(a_i)\, e_j$.
3. *For any $i \neq j$,*

$$\begin{aligned} R_{ijij} &= \left\langle R_{e_i e_j} e_i, e_j \right\rangle \\ &= -\frac{1}{a_j} e_i e_i(a_j) - \frac{1}{a_i} e_j e_j(a_i) - \sum_{l \neq i,j} \left(\frac{1}{a_i} e_l(a_i) \frac{1}{a_j} e_l(a_j) \right) \end{aligned}$$

$$R_{ijik} = \left\langle R_{e_i e_j} e_i, e_k \right\rangle \; = \; -\frac{1}{a_i} e_j e_k(a_i) + \frac{1}{a_i} e_j(a_i) \frac{1}{a_j} e_k(a_j).$$

4. *If i, j, k, l are all distinct, $\left\langle R_{e_i e_j} e_k, e_l \right\rangle = R_{ijkl} = 0$.*

The algebra of curvature tensors

The space of all such *algebraic curvature operators* in dimension n is the space $\mathcal{R}_n$ of symmetric homomorphisms of $\Lambda_2(\mathbb{R}^n) = \Lambda_2(T_*(M, x))$, with the standard induced inner product on $\Lambda_2(\mathbb{R}^n)$, defined by $\langle R(v_1 \wedge v_2), v_3 \wedge v_4 \rangle = \left\langle R_{(v_1, v_2)} v_3, v_4 \right\rangle$, which, due to the identities of the curvature tensor such as $R_{XY} = -R_{YX}$, gives a linear operator

$$R : \Lambda_2(\mathbb{R}^n) \quad \rightarrow \quad \Lambda_2(\mathbb{R}^n)$$

which is symmetric.

Decomposition

$SO(n)$ operates on this space induced from its natural action on $\mathbb{R}^n$, which, following Weyl, decomposes $\mathcal{R}_n$ into an orthogonal direct sum of invariant subspaces:

$$\mathcal{R}_n \quad = \quad \mathcal{I} \oplus \mathcal{RIC}_0 \oplus \mathcal{W} \oplus \mathcal{S},$$

where:

- $\mathcal{I}$ are all multiples of the identity operator,
- for $\rho : \mathcal{R}_n \to \mathbb{R}^n \circ \mathbb{R}^n$ the Ricci contraction $\langle \rho(R) v, w \rangle = \sum \langle R(v \wedge e_i), w \wedge e_i \rangle$,$\mathcal{RIC}_0 := \ker(\rho)^\perp \cap \mathcal{I}^\perp$ corresponds to the trace-free portion of the Ricci tensor,
- $\mathcal{S} := \ker(b)^\perp$ are those tensors orthogonal to the kernel of the Bianchi map $b : \mathcal{R}_n \to \mathcal{R}_n$ defined by $b(R)_{(v_1,v_2)} v_3 = R_{(v_1,v_2)} v_3 + R_{(v_2,v_3)} v_1 + R_{(v_3,v_1)} v_2$, and
- $\mathcal{W} = \ker(\rho) \cap \ker(b)$ is the Weyl tensor component.

Dimension 4

In dimension 4 this decomposition has a particularly simple form, due to Atiyah-Hitchen-Singer and LeBrun. In that dimension the Hodge star operator $\star : \Lambda_2\left(\mathbb{R}^4\right) \to \Lambda_2\left(\mathbb{R}^4\right)$ is itself a curvature operator; in particular it is a basis of the space $\mathcal{S}$ of operators orthogonal to $\ker(b)$. The operator $\star$ is defined on any oriented orthonormal frame $\{e_1, e_2, e_3, e_4\}$ by $\star(e_1 \wedge e_2) = e_3 \wedge e_4$, $\star(e_1 \wedge e_3) = -e_2 \wedge e_4$, and $\star(e_1 \wedge e_4) = e_2 \wedge e_3$. Since $\star^2 = Id$, it decomposes $\Lambda_2\left(\mathbb{R}^4\right)$ into two 3-dimensional subspaces $\Lambda_2^+\left(\mathbb{R}^4\right)$ and $\Lambda_2^-\left(\mathbb{R}^4\right)$ consisting of the ± 1-eigenspaces of $\star$,

$$\Lambda_2^+\left(\mathbb{R}^4\right) = \mathbf{Span}\left\{e_1 \wedge e_2 + e_3 \wedge e_4, e_1 \wedge e_3 - e_2 \wedge e_4, e_1 \wedge e_4 + e_2 \wedge e_3\right\}$$

and

$$\Lambda_2^-\left(\mathbb{R}^4\right) = \mathbf{Span}\left\{e_1 \wedge e_2 - e_3 \wedge e_4, e_1 \wedge e_3 + e_2 \wedge e_4, e_1 \wedge e_4 - e_2 \wedge e_3\right\}.$$

For a curvature form R on an oriented 4-manifold M, its Weyl tensor $W(R) \in \mathcal{W}$, decomposes as $W(R) = W^+(R) + W^-(R)$, where $W^+(R) : \Lambda_2^+\left(\mathbb{R}^4\right) \to \Lambda_2^+\left(\mathbb{R}^4\right)$ and $W^-(R) : \Lambda_2^-\left(\mathbb{R}^4\right) \to \Lambda_2^-\left(\mathbb{R}^4\right)$ are the orthogonal projections of $W(R)$ onto the indicated subspaces. A manifold is self-dual if $W^-(R) = 0$, and anti-delf-dual if $W^+(R) = 0$.

Curvature structure

Given any oriented frame $\{e_1, e_2, e_3, e_4\}$, we will use the *adapted frame*, writing $\Lambda_2\left(\mathbb{R}^4\right) = \Lambda_2^+\left(\mathbb{R}^4\right) \oplus \Lambda_2^-\left(\mathbb{R}^4\right)$,

$$\left\{ \frac{1}{\sqrt{2}}\left(e_1 \wedge e_2 + e_3 \wedge e_4\right), \frac{1}{\sqrt{2}}\left(e_1 \wedge e_3 - e_2 \wedge e_4\right), \frac{1}{\sqrt{2}}\left(e_1 \wedge e_4 + e_2 \wedge e_3\right), \right.$$
$$\left. \frac{1}{\sqrt{2}}\left(e_1 \wedge e_2 - e_3 \wedge e_4\right), \frac{1}{\sqrt{2}}\left(e_1 \wedge e_3 + e_2 \wedge e_4\right), \frac{1}{\sqrt{2}}\left(e_1 \wedge e_4 - e_2 \wedge e_3\right) \right\}.$$

Proposition

Singer-Thorpe, AHS, LeBrun *Any $R \in \mathcal{R}_4$ satisfying the Bianchi identity decomposes into 3×3 blocks:*

$$R = \left[\begin{array}{c|c} W^+(R) + \frac{r}{12} Id & s\left(\rho\left(R_0\right)\right) \\ \hline s\left(\rho\left(R_0\right)\right)^T & W^-(R) + \frac{r}{12} Id \end{array}\right]$$

with respect to the adapted frame of any orthonormal frame $\{e_1, \ldots, e_4\}$, where $r = 2\mathrm{tr}(R)$ is the scalar curvature $r = \mathrm{tr}(\rho(R))$, $\rho(R_0) = \rho(R) - \frac{r}{4} I$, and s is the right inverse of the Ricci projection ρ.

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Surfaces

On an oriented surface Σ, isothermal coordinates, coordinates so that the metric is of the form $g = e^{\lambda}\left(dx^2 + dy^2\right)$ always exist. The beauty of these coordinates is that they also give the notion of complex structure to the surface; the tangent planes can be thought of as the complex numbers, that is, there is a map $J : T_*\left(\Sigma, x\right) \to T_*\left(\Sigma, x\right)$ rotating each vector by $+\frac{\pi}{2}$ (or multiplying by i). This map J is called an *almost-complex* structure, and satisfies $J^2 = -Id$. But, in isothermal coordinates charts, that rotation angle is preserved by the differential of those coordinates, which is exactly the Cauchy-Riemann equations, so the coordinate transformations between two such charts is holomorphic,

$$\left(\phi \circ \psi^{-1}\right)_* (JX) = J\left(\phi \circ \psi^{-1}\right)_* (X),$$

or the almost-complex structure is *integrable*.

Higher dimensions

It is reasonable to ask whether these ideas always extend to higher dimensions. That story becomes complicated. In dimension 3 or higher, isothermal coordinates (called *locally conformally flat* in higher dimensions) don't usually exist. Dimension 3 is subtle, but in dimension 4 or higher the obstruction to locally conformally flat metrics is the *Weyl tensor*.

In higher even dimensions, the existence of an almost-complex structure is not guaranteed (S^4 does not have one), and integrability, the existence of holomorphic coordinate changes, is even more restrictive. Spaces which are complex varieties inside complex projective space always are *complex manifolds*, as are spaces like $S^3 \times S^1$.

Kähler manifolds

A rich and elegant collection of complex manifolds, called *Kähler manifolds*, are complex manifolds with a Riemannian metric $ds^2 = \langle , \rangle$ which is compatible with the complex structure, or is *Hermitian*, if $\langle JX, JY \rangle = \langle X, Y \rangle$, that is, if J is orthogonal. In addition, we want the almost-complex structure to be integrable, and we add one more condition, called the Kähler condition, that the Kähler form

$$\omega(X, Y) \;=\; \langle X, JY \rangle$$

is a closed 2-form. These two conditions are equivalent to the the almost complex structure tensor being *parallel*, $\nabla J = 0$. Kähler manifolds include complex subvarieties of $\mathbb{CP}^n$, so we think of them as the most natural complex manifolds.

Complex structure's multiple realities

If M is a Kähler manifold, the complex-structure tensor J, given by $J(e_i) = a_{ij} e_j$, is an isometry on each $T_*(M, m) \cong \mathbb{R}^4$ satisfying $J^2 = -Id$, so that J is also skew-symmetric. Identifying $o(4)$ with $\Lambda_2(\mathbb{R}^4)$, J becomes the bivector $I \in \Lambda_2(\mathbb{R}^4)$, the metric dual of the Kähler form, $I = \sum_{i<j} a_{ij} e_i \wedge e_j$. In dimension 4, the orientation is consistent with the complex structure if $I \in \Lambda_2^+(\mathbb{R}^4)$, so for any oriented frame $\{e_1, e_2, e_3, e_4\}$, $I = a_{12}(e_1 \wedge e_2 + e_3 \wedge e_4) + a_{13}(e_1 \wedge e_3 - e_2 \wedge e_4) + a_{14}(e_1 \wedge e_4 + e_2 \wedge e_3)$ with $a_{12}^2 + a_{13}^2 + a_{14}^2 = 1$. As an operator on the tangent space, then, in dimension 4,

$$
\begin{aligned}
J(e_1) &= a_{12} e_2 + a_{13} e_3 + a_{14} e_4 \\
J(e_2) &= -a_{12} e_1 + a_{14} e_3 - a_{13} e_4 \\
J(e_3) &= -a_{13} e_1 - a_{14} e_2 + a_{12} e_4 \\
J(e_4) &= -a_{14} e_1 + a_{13} e_2 - a_{12} e_3 .
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J(e_4) &= -a_{14} e_1 + a_{13} e_2 - a_{12} e_3.
\end{aligned}$$

J and $\star$ as curvature operators

J can be extended to an algebraic curvature operator
$J:\Lambda_2\left(\mathbb{R}^4\right)\to\Lambda_2\left(\mathbb{R}^4\right)$ by $J\left(v\wedge w\right):=J\left(v\right)\wedge J\left(w\right)$.
$J:\Lambda_2\left(\mathbb{R}^4\right)\to\Lambda_2\left(\mathbb{R}^4\right)$ Like the Hodge star operator $\star$, J is idempotent, $J^2=Id$, with a 4-dimensional eigenspace for the eigenvalue 1, and a 2-dimensional eigenspace for the eigenvalue -1. Since $J\star=\star J$, J and $\star$ are simultaneously diagonalizable. $J|_{\Lambda_2^-\left(\mathbb{R}^4\right)}:\Lambda_2^-\left(\mathbb{R}^4\right)\to\Lambda_2^-\left(\mathbb{R}^4\right)$ is the identity, and $J\left(I\right)=I$, where I is as above. The orthogonal complement of I within $\Lambda_2^+\left(\mathbb{R}^4\right)$ is the (-1)-eigenspace, spanned by
$a_{13}\left(e_1\wedge e_2+e_3\wedge e_4\right)-a_{12}\left(e_1\wedge e_3-e_2\wedge e_4\right)$ and
$a_{14}\left(e_1\wedge e_2+e_3\wedge e_4\right)-a_{12}\left(e_1\wedge e_4+e_2\wedge e_3\right)$.

Kähler curvature operators

If R is a curvature operator corresponding to a Kähler 4-manifold, then $b(R) = 0$ and

$$RJ = JR = R,$$

so that in the (-1)-eigenspace of J, $I^{\perp} \cap \Lambda_2^+ (\mathbb{R}^4) = \{\xi | \, J\xi = -\xi\}$, $-R(\xi) = R(J\xi) = R(\xi)$, so $R(\xi) = 0$.

The conditions for R to be Kähler then become simply

$$\begin{aligned} 0 &= R\left(a_{12}\left(e_1 \wedge e_3 - e_2 \wedge e_4\right) - a_{13}\left(e_1 \wedge e_2 + e_3 \wedge e_4\right)\right), \text{ and} \\ 0 &= R\left(a_{12}\left(e_1 \wedge e_4 + e_2 \wedge e_3\right) - a_{14}\left(e_1 \wedge e_2 + e_3 \wedge e_4\right)\right), \end{aligned}$$

Kähler-adapted frame

Re-write the curvature tensor R with respect to this frame:

$$\left\{ \frac{1}{\sqrt{2}} I, \frac{a_{13}}{\sqrt{2\left(a_{12}^2 + a_{13}^2\right)}} \left(e_1 \wedge e_2 + e_3 \wedge e_4\right) - \frac{a_{12}}{\sqrt{2\left(a_{12}^2 + a_{13}^2\right)}} \left(e_1 \wedge e_3 - e_2 \wedge \right.\right.$$

$$\frac{a_{14}}{\sqrt{2\left(a_{12}^2 + a_{14}^2\right)}} \left(e_1 \wedge e_2 + e_3 \wedge e_4\right) - \frac{a_{12}}{\sqrt{2\left(a_{12}^2 + a_{14}^2\right)}} \left(e_1 \wedge e_4 + e_2 \wedge e_3\right),$$

$$\left. \frac{1}{\sqrt{2}} \left(e_1 \wedge e_2 - e_3 \wedge e_4\right), \frac{1}{\sqrt{2}} \left(e_1 \wedge e_3 + e_2 \wedge e_4\right), \frac{1}{\sqrt{2}} \left(e_1 \wedge e_4 - e_2 \wedge e_3\right) \right\}.$$

The second and third vectors are a basis of the -1-eigenspace of J.

Notation

Set, for all i, j

$$\rho_{ij} \quad = \rho(R)_{ij} = \sum_{k \neq i,j} R_{ikjk},$$

the components of the Ricci tensor, and, with $k_1 < k_2$ and $i < j$

$$\widehat{\rho}_{ij} \quad = \quad R_{ik_1jk_1} - R_{ik_2jk_2},$$

which correspond to the components of the Weyl tensor (in dimension 4).

R

$$R = \left[\begin{array}{ccc|ccc} \frac{r}{4} & 0 & 0 & \frac{1}{2a_{14}}\left(\rho_{24}-\rho_{13}\right) & \frac{1}{2a_{12}}\left(\rho_{23}-\rho_{14}\right) & \frac{1}{2a_{13}}\left(\rho_{34}-\rho_{12}\right) \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline & & & \frac{a_{12}^2 r}{4} - 2R_{1234} & \frac{1}{2}\left(\hat{\rho}_{23}-\hat{\rho}_{14}\right) & \frac{1}{2}\left(\hat{\rho}_{24}+\hat{\rho}_{13}\right) \\ & T & & \frac{1}{2}\left(\hat{\rho}_{23}-\hat{\rho}_{14}\right) & \frac{a_{13}^2 r}{4} + 2R_{1324} & \frac{1}{2}\left(\hat{\rho}_{34}-\hat{\rho}_{12}\right) \\ & & & \frac{1}{2}\left(\hat{\rho}_{24}+\hat{\rho}_{13}\right) & \frac{1}{2}\left(\hat{\rho}_{34}-\hat{\rho}_{12}\right) & \frac{a_{14}^2 r}{4} - 2R_{2314} \end{array}\right]$$

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Fubini-Study metric on $\mathbb{CP}^2$

As an example, for any frame, the curvature of the standard Fubini-Study metric of complex projective 2-space has the form

$$
\begin{aligned}
R &= \left[\begin{array}{c|ccc} 6 & 0 & 0 & 0 \\ \hline 0 & 6a_{12}^2 - 2R_{1234} & 0 & 0 \\ 0 & 0 & 6a_{13}^2 + 2R_{1324} & 0 \\ 0 & 0 & 0 & 6a_{14}^2 - 2R_{2314} \end{array}\right] \\
0 &= \left[\begin{array}{c|ccc} 6 & 0 & 0 & 0 \\ \hline 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{array}\right].
\end{aligned}
$$

Product metric

A simple example of a space which supports orthogonal coordinates is a (local) Riemannian product of two Riemann surfaces Σ_1 and Σ_2 of Gaussian curvatures r_1 and r_2, respectively. With respect to the obvious unitary frame with $\{e_1, e_2\}$, resp., $\{e_3, e_4\}$ being frames of the two surfaces,

$$R = \left[\begin{array}{c|ccc} \frac{r_1+r_2}{2} & \frac{r_1-r_2}{2} & 0 & 0 \\ \hline \frac{r_1-r_2}{2} & \frac{r_1+r_2}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

Orthogonal coordinates and Kähler 4-manifolds

Now we assume that M^4 is a Kähler manifold, and that it supports orthogonal coordinates. The only thing that changes in the expression of the curvature operator, using a frame associated to the orthogonal coordinates, is that the terms R_{ijkl} vanish when all four indices are distinct.

$$R = \left[\begin{array}{c|ccc} \frac{r}{4} & \frac{1}{2a_{14}}\left(\rho_{24}-\rho_{13}\right) & \frac{1}{2a_{12}}\left(\rho_{23}-\rho_{14}\right) & \frac{1}{2a_{13}}\left(\rho_{34}-\rho_{12}\right) \\ \hline & \frac{a_{12}^2 r}{4} & \frac{1}{2}\left(\hat{\rho}_{23}-\hat{\rho}_{14}\right) & \frac{1}{2}\left(\hat{\rho}_{24}+\hat{\rho}_{13}\right) \\ T & \frac{1}{2}\left(\hat{\rho}_{23}-\hat{\rho}_{14}\right) & \frac{a_{13}^2 r}{4} & \frac{1}{2}\left(\hat{\rho}_{34}-\hat{\rho}_{12}\right) \\ & \frac{1}{2}\left(\hat{\rho}_{24}+\hat{\rho}_{13}\right) & \frac{1}{2}\left(\hat{\rho}_{34}-\hat{\rho}_{12}\right) & \frac{a_{14}^2 r}{4} \end{array}\right]$$

However, beyond these curvature conditions, the condition that the Kähler form be parallel, $\nabla J = 0$, or, equivalently,

$$\nabla_X (JY) = J \nabla_X Y$$

is stronger.

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$\nabla J = 0$

If M has orthogonal coordinates as above, with frame $e_i = \frac{1}{a_i}\frac{\partial}{\partial x_i}$, $\nabla J = 0$ is equivalent to

$$
\begin{aligned}
a_1 e_1\left(\begin{bmatrix} a_{12} \\ a_{13} \\ a_{14} \end{bmatrix}\right) &= \begin{bmatrix} e_2(a_1) \\ e_3(a_1) \\ e_4(a_1) \end{bmatrix} \times \begin{bmatrix} a_{12} \\ a_{13} \\ a_{14} \end{bmatrix} \\
a_2 e_2\left(\begin{bmatrix} a_{12} \\ a_{13} \\ a_{14} \end{bmatrix}\right) &= \begin{bmatrix} -e_1(a_2) \\ -e_4(a_2) \\ e_3(a_2) \end{bmatrix} \times \begin{bmatrix} a_{12} \\ a_{13} \\ a_{14} \end{bmatrix} \\
a_3 e_3\left(\begin{bmatrix} a_{12} \\ a_{13} \\ a_{14} \end{bmatrix}\right) &= \begin{bmatrix} e_4(a_3) \\ -e_1(a_3) \\ -e_2(a_3) \end{bmatrix} \times \begin{bmatrix} a_{12} \\ a_{13} \\ a_{14} \end{bmatrix} \\
a_4 e_4\left(\begin{bmatrix} a_{12} \\ a_{13} \\ a_{14} \end{bmatrix}\right) &= \begin{bmatrix} -e_3(a_4) \\ e_2(a_4) \\ -e_1(a_4) \end{bmatrix} \times \begin{bmatrix} a_{12} \\ a_{13} \\ a_{14} \end{bmatrix}.
\end{aligned}
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Uniqueness of J

If a nontrivial M supports orthogonal coordinates, then, of course, there are several such coordinate systems, which overlap. A coordinate change from one orthogonal coordinate system to another would not seem to preserve in any way the complex structure. However, on a non-flat manifold the complex structure will be preserved (The change of coordinates may not be holomorphic, but is so up to "scaling.")

Definition

A hyperkähler manifold is a Riemannian manifold with 3 integrable almost complex structures I, J, and $K = IJ$, for which the metric is Kähler with respect to each, and so that $I^2 = J^2 = K^2 = -Id$. (E. Calabi) Think of the quaternions.

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If there are two distinct complex structure tensors J, K on an orthogonal coordinate chart that both correspond to Kähler structures in the neighborhood, then the metric will be hyperkähler on that chart.

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Proof.

If both J and K satisfy the conditions, with K given by $\begin{bmatrix} b_{12} \\ b_{13} \\ b_{14} \end{bmatrix}$, then, using those equations, it is easy to show that $\begin{bmatrix} a_{12} \\ a_{13} \\ a_{14} \end{bmatrix} \cdot \begin{bmatrix} b_{12} \\ b_{13} \\ b_{14} \end{bmatrix}$ is constant. Then, you can construct yet another Kähler structure L given by $\begin{bmatrix} c_{12} \\ c_{13} \\ c_{14} \end{bmatrix} = \left(\begin{bmatrix} b_{12} \\ b_{13} \\ b_{14} \end{bmatrix} - \left(\begin{bmatrix} a_{12} \\ a_{13} \\ a_{14} \end{bmatrix} \cdot \begin{bmatrix} b_{12} \\ b_{13} \\ b_{14} \end{bmatrix} \right) \begin{bmatrix} a_{12} \\ a_{13} \\ a_{14} \end{bmatrix} \right) \Big/ \|\|$, and the vector $\begin{bmatrix} a_{12} \\ a_{13} \\ a_{14} \end{bmatrix} \times \begin{bmatrix} c_{12} \\ c_{13} \\ c_{14} \end{bmatrix}$ will also correspond to a Kähler structure, corresponding to JL, which then will be hyperkähler. □

Hyperkähler

Theorem

No Kähler metric on a manifold M^4 supporting orthogonal coordinates is hyperkähler, unless it is flat. Thus no K3 surface supports orthogonal coordinates.

Corollary

If M^4 is a nowhere flat Kähler manifold with orthogonal coordinates, the complex structure is uniquely determined on each coordinate chart.

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Theorem

If M is a 4-dimensional Kähler manifold with orthogonal coordinates, and if the coefficients a_{ij} of the complex structure tensor J with respect to the associated frame $\{e_1, e_2, e_3, e_4\}$ are constant, then necessarily M is either locally a product of Riemann surfaces, or it is flat.

Proof.

Gauduchon and Moroiano show this result in a special case ($a_{12} = a_{13} = a_{14} = \frac{1}{\sqrt{3}}$); a similar proof holds for any constants. □

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Self dual

Theorem

If M^4 is a self-dual Kähler 4-manifold which supports orthogonal coordinates, then M is flat, or a product $\Sigma_r \times \Sigma_{-r}$ of constant curvature Riemann surfaces.

Proof.

Assuming that M is self-dual, then with respect to a frame $\{e_1, e_2, e_3, e_4\}$ from an orthogonal coordinate chart,

$$\begin{aligned} 0 &= W^-(R) \\ &= \begin{bmatrix} \frac{r}{4}\left(a_{12}^2 - \frac{1}{3}\right) & \frac{1}{2}\left(\hat{\rho}_{23} - \hat{\rho}_{14}\right) & \frac{1}{2}\left(\hat{\rho}_{24} + \hat{\rho}_{13}\right) \\ \frac{1}{2}\left(\hat{\rho}_{23} - \hat{\rho}_{14}\right) & \frac{r}{4}\left(a_{13}^2 - \frac{1}{3}\right) & \frac{1}{2}\left(\hat{\rho}_{34} - \hat{\rho}_{12}\right) \\ \frac{1}{2}\left(\hat{\rho}_{24} + \hat{\rho}_{13}\right) & \frac{1}{2}\left(\hat{\rho}_{34} - \hat{\rho}_{12}\right) & \frac{r}{4}\left(a_{14}^2 - \frac{1}{3}\right) \end{bmatrix} \end{aligned}$$

so that, either $r = 0$, or $a_{12}^2 = a_{13}^2 = a_{14}^2 = \frac{1}{3}$. In the first case, the manifold must be conformally flat, thus either flat or a product of Riemann surfaces with opposite constant curvatures. The conclusion then follows from the previous result. □

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$\mathbb{CP}^2$

Corollary

$\mathbb{CP}^2$ with the Fubini-Study metric does not support orthogonal coordinates.

Proof.

This is the result of Gauduchon and Moroianu, and follows by the fact that $\mathbb{CP}^2$ is indeed self-dual. □

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$\mathbb{CP}^2$

Fact

$\mathbb{CP}^2$ with the Fubini-Study metric does admit a frame $\{e_1, e_2, e_3, e_4\}$ so that $R_{1234} = R_{1324} = R_{1423} = 0$.

Proof.

Start with a unitary frame $\{u_1, u_2 = Ju_1, u_3, u_4 = Ju_2\}$. Then

$$\begin{aligned} e_1 &= u_1 \\ e_2 &= \frac{1}{\sqrt{3}}u_2 + \frac{1}{\sqrt{2}}u_3 + \frac{1}{\sqrt{6}}u_4 \\ e_3 &= \frac{1}{\sqrt{3}}u_2 - \frac{1}{\sqrt{2}}u_3 + \frac{1}{\sqrt{6}}u_4 \\ e_4 &= \frac{1}{\sqrt{3}}u_2 - \frac{\sqrt{2}}{\sqrt{3}}u_4. \end{aligned}$$

This gives a frame $\{e_1, e_2, e_3, e_4\}$ on $\mathbb{CP}^2$, which satisfies the conditions $a_{12} = a_{13} = a_{14} = \frac{1}{\sqrt{3}}$ for the complex structure tensor with respect to that frame, and $R_{ijkl} = 0$ whenever all indices are distinct.

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Thanks.

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