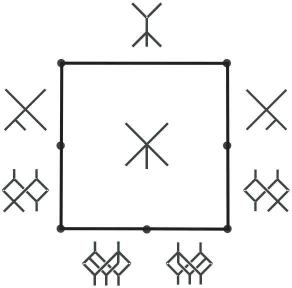
Gerstenhaber-Schack Bialgebras

Presented by Ron Umble
Professor Emeritus, MU

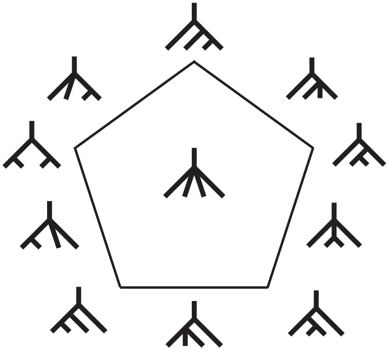
TETRAHEDRAL GEOMETRY-TOPOLOGY SEMINAR

October 4, 2024



The Associahedron K_n

Let $n \geq 2$. The associahedron K_n is an $(n-2)$-dimensional contractible polytope constructed by J. Stasheff (1963) whose faces are indexed by up-rooted planar trees with n leaves



The Associahedron K_4

A_k-Algebras

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- *Strict A_3-algebras are associative*

The Coassociahedron K^n

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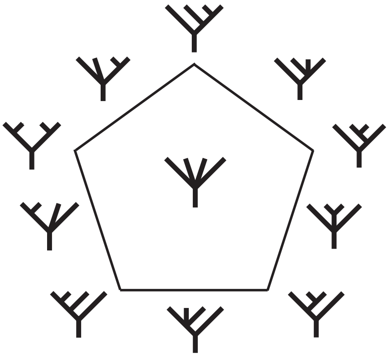

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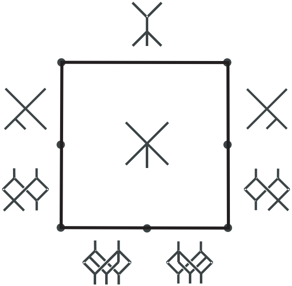
- *Strict A_3-coalgebras are coassociative*

The Biassociahedron KK_m^n

Let $m, n \geq 1$ and $m + n \geq 3$. The biassociahedron KK_m^n is an $(m + n - 3)$-dimensional contractible polytope constructed by S. Saneblidze and U (2022) with faces indexed by m-in/n-out upward-directed graphs



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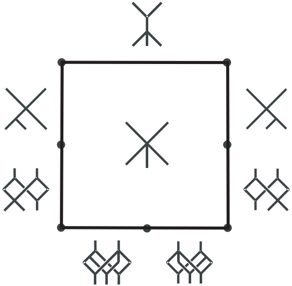

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- $KK_n^1 = K_n \cong K^n = KK_1^n$

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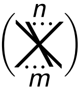
  where $\alpha_m^n\Big($ 

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- *A_k-algebras are A_{k+1}-bialgebras with $\omega_m^n = 0$ for all $n > 1$*
- *Strict A_4-bialgebras are graded Hopf algebras*

Structure Maps and Relations

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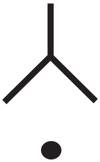
- Combinatorics of KK_m^n encode the *structure relation*

$\nabla \omega_m^n = (\alpha_m^n \circ \partial)$ ()

The Differential is a Biderivation

KK_2^1 :



$$\nabla \mu = d\mu + \mu(d \otimes \mathbf{1} + \mathbf{1} \otimes d) = 0$$

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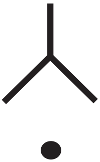

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Dually, KK_1^2 is a point and d is a coderivation

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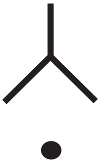

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- *The differential in an A_k-bialgebra is a biderivation*

Homotopy Biassociativity

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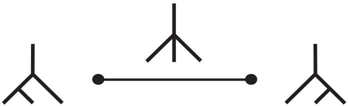

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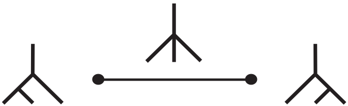

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$$\mu(\mu \otimes \mathbf{1}) = \mu(\mathbf{1} \otimes \mu)$$

$$(\Delta \otimes \mathbf{1})\Delta = (\mathbf{1} \otimes \Delta)\Delta$$

Homotopy Compatibility

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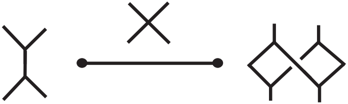

$$\nabla \omega_2^2 = \Delta\mu + (\mu \otimes \mu)\sigma_{2,2}(\Delta \otimes \Delta)$$

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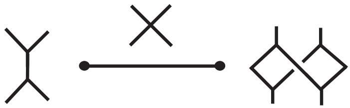

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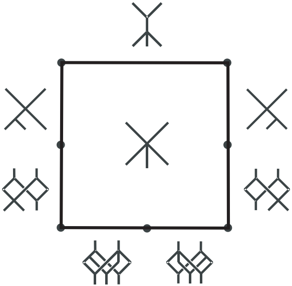
Decoding the KK_3^2 Structure Relation



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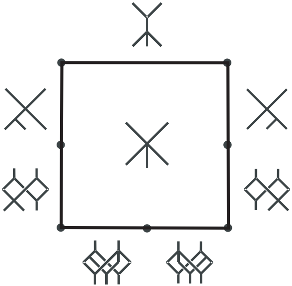

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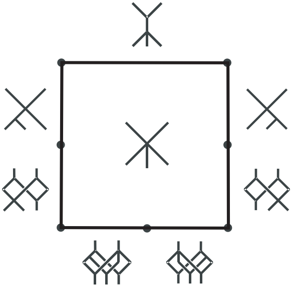

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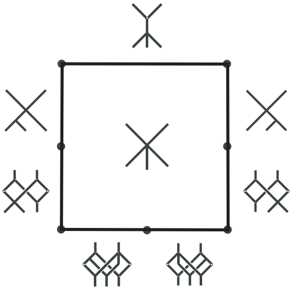

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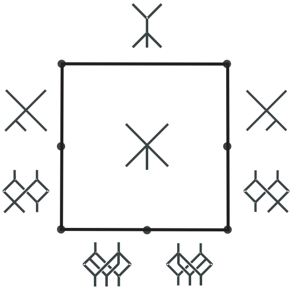

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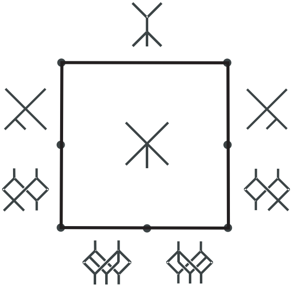

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- *All structure relations are decoded in a similar way*

A Topological Application

- Let $a_i \in H^i\left(S^i;\mathbb{Z}_2\right)$ and $b \in H^3\left(\Sigma\mathbb{C}P^2;\mathbb{Z}_2\right)$

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$$\begin{array}{ccccc} K(\mathbb{Z}_2,4) & \longrightarrow & X & \longrightarrow & \mathcal{L}K(\mathbb{Z}_2,5) \\ & & p\downarrow & & \downarrow \\ & & \left(S^2\times S^3\right)\vee\Sigma\mathbb{C}P^2 & \xrightarrow{f} & K(\mathbb{Z}_2,5) \\ & & a_2a_3+Sq^2b & \underset{f^*}{\longleftarrow} & \iota_5 \end{array}$$

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- $H^*(X;\mathbb{Z}_2)=\{1,a_2,a_3,b,a_2a_3=Sq^2b,\ldots\}$
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- Let μ_H and Δ_H be the induced operations on $H := H^*(BA)$

A Computational Model for $H^*(\Omega X)$

- $A := H^*(X; \mathbb{Z}_2)$ is a graded commutative algebra
- The bar construction BA with standard differential d_{BA} and free coproduct Δ is a dg coalgebra
- There is a perturbation μ of the shuffle product

$$sh([a] \otimes [b]) := [a|b] + [b|a],$$

which acts as the shuffle product except

$$\mu([b] \otimes [b]) = [a_2 a_3] = d_{BA}[a_2|a_3]$$

- Let μ_H and Δ_H be the induced operations on $H := H^*(BA)$
- $(H, \mu_H, \Delta_H) \approx H^*(\Omega X; \mathbb{Z}_2)$ as graded Hopf algebras

Transfer of dg Hopf Algebra Structure

The Transfer Theorem (Saneblidze-U 2011)

For H as above, a cocycle-selecting map $g : H \to BA$ induces an A_∞-bialgebra structure

$$\omega = \{\omega_m^n : H^{\otimes m} \to H^{\otimes n}\}$$

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- $(H, \mu_H, \Delta_H, \omega_3^1, \omega_2^2)$ is a "Gerstenhaber-Schack bialgebra"

The Gerstenhaber-Schack Complex

The **Gerstenhaber-Schack** (G-S) **Complex** of a dg Hopf algebra (H, d, μ, Δ) is the triple complex

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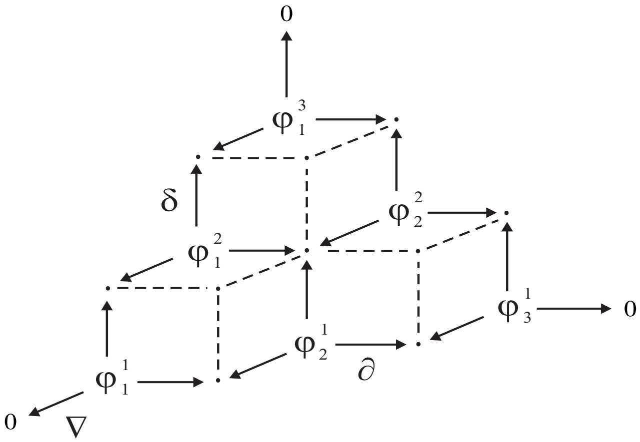
- The r^{th} G-S cohomology group in degree p

$$H_{GS}^{r,p}(H; H) := H^*\left(C_{GS}^{r,p}(H, H), D\right)$$

A 2-Cocycle with $m + n \leq 4$



$$D(\varphi_1^1 + \varphi_2^1 + \varphi_1^2 + \varphi_3^1 + \varphi_2^2 + \varphi_1^3) = 0$$

The KK_3^2 Structure Relation (revisited)

$$\nabla\omega_3^2 = \omega_2^2(\mu\otimes\mathbf{1}+\mathbf{1}\otimes\mu)+(\mu\otimes\mu)\sigma_{2,2}(\omega_2^2\otimes\Delta+\Delta\otimes\omega_2^2)$$
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By definition

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- *Other relations with $m+n=5$ have similar representations*

Structure Relations with $m + n = 5$

$$KK_1^4 : \qquad \nabla\omega_1^4 = \delta\omega_1^3 \qquad \overset{\nabla\omega=0}{\Rightarrow} \qquad \delta\omega_1^3 = 0$$

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- *In strict A_5-bialgebras*

$$D(\omega_3^1 + \omega_2^2 + \omega_1^3) = \partial(\omega_3^1 + \omega_2^2 + \omega_1^3) + \delta(\omega_3^1 + \omega_2^2 + \omega_1^3)$$
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- *In strict A_5-bialgebras*

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\begin{aligned}
D(\omega_3^1 + \omega_2^2 + \omega_1^3) &= \partial(\omega_3^1 + \omega_2^2 + \omega_1^3) + \delta(\omega_3^1 + \omega_2^2 + \omega_1^3) \\
&= \delta\omega_1^3 + (\partial\omega_1^3 + \delta\omega_2^2) + (\partial\omega_2^2 + \delta\omega_3^1) + \partial\omega_3^1 = 0
\end{aligned}
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- $\omega_3^1, \omega_2^2, \omega_1^3$ *satisfy the strict A_5-bialgebra structure relations iff*

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- $\omega_3^1, \omega_2^2, \omega_1^3$ *satisfy the strict A_5-bialgebra structure relations iff* $\omega_3^1 + \omega_2^2 + \omega_1^3$ *is the deg -1 component of a strict 2-cocycle*

The Degree -1 Component of a Strict 2-Cocycle

$$
\begin{array}{ccccccc}
\delta\omega_1^3 = 0 & & & & & & \\
\uparrow & & & & & & \\
\omega_1^3 & \longrightarrow & \partial\omega_1^3 + \delta\omega_2^2 = 0 & & & & \\
 & & \uparrow & & & & \\
 & & \omega_2^2 & \longrightarrow & \partial\omega_2^2 + \delta\omega_3^1 = 0 & & \\
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$$D(\omega_3^1 + \omega_2^2 + \omega_1^3) = 0$$

Gerstenhaber-Schack Bialgebras

An A_4-bialgebra $(H, \mu, \Delta, \omega_3^1, \omega_2^2, \omega_1^3)$ is a **Gerstenhaber-Schack bialgebra** if

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- $(H, \mu_H, \Delta_H, \omega_3^1, \omega_2^2)$ *is a G-S bialgebra*

G-S Extensions of Graded Hopf Algebras

- A **G-S extension** of a graded Hopf algebra (H, μ, Δ) is a G-S bialgebra $(H, \mu, \Delta, \omega := \{\omega_3^1, \omega_2^2, \omega_1^3\})$

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A potential remedy might be to extend the G-S complex to a multicomplex with additional differentials defined in terms of the higher order operations.

Finally, it would be nice to have a family of spaces X_k whose cohomology admits an A_k but not an A_{k+1}-bialgebra structure. I'll leave this problem for homework!

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THANK YOU!