Comparing Diagonals on the Associahedra

Joint work with Samson Saneblidze
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TETRAHEDRAL GEOMETRY/TOPOLOGY SEMINAR

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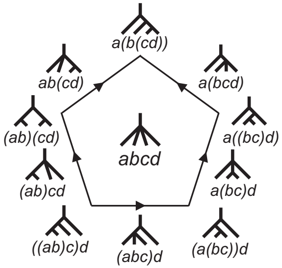

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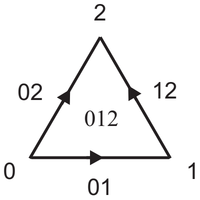
- Δ_X induces a chain map of cellular chains

  $$\Delta_X : (C_*(X)\partial) \to (C_*(X) \otimes C_*(X), \partial \otimes \mathbf{1} + \mathbf{1} \otimes \partial)$$

  called a **diagonal** on $(C_*(X), \partial)$

Alexander-Whitney and Serre Diagonals

$$\Delta_S(01\cdots n) = \sum_{i=0}^{n} 01\cdots i \times i\cdots n;$$



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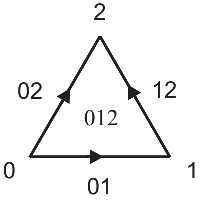
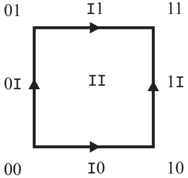
$$\Delta_S(01\cdots n) = \sum_{i=0}^{n} 01\cdots i \times i\cdots n; \quad \Delta_{\mathtt{I}}(\mathtt{I}^n) = \sum_{(u_1,\ldots,u_n)\in\{0,\mathtt{I}\}^{\times n}} \pm u_1\cdots u_n \times u'_1\cdots u'_n$$

$$(0' = \mathtt{I}, \mathtt{I}' = 1)$$





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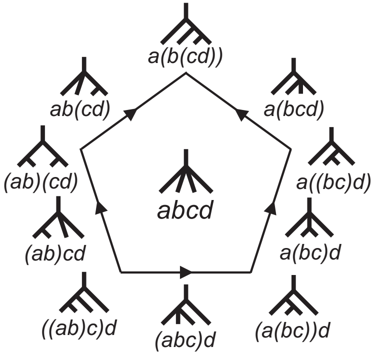
$$\Delta_{\mathtt{I}}\left(\mathtt{I}^2\right) = 00 \times \mathtt{II} - 0\mathtt{I} \times \mathtt{I}1 + \mathtt{I}0 \times 1\mathtt{I} + \mathtt{II} \times 11$$

The Associahedron K_n

K_n is an $(n-2)$-dimensional contractible polytope constructed by J. Stasheff (1963) whose faces are indexed by planar rooted trees (PRTs) with n leaves



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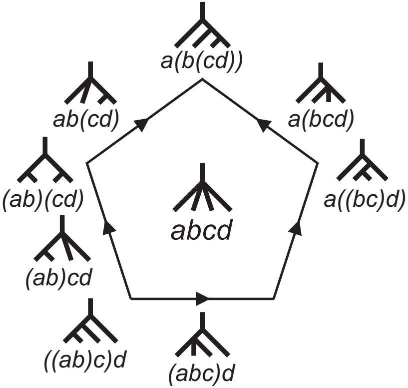

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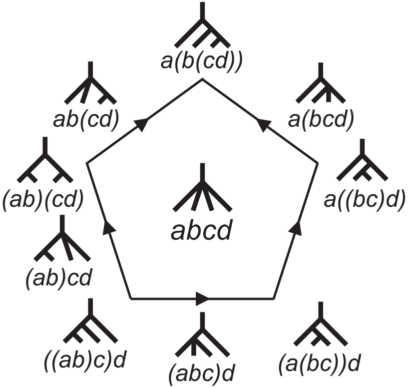

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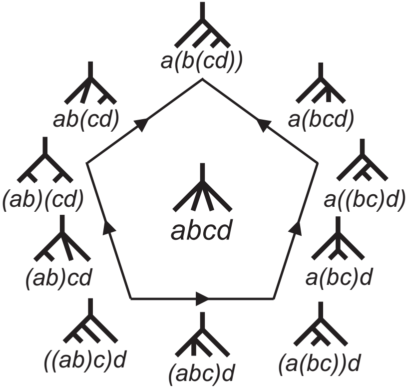

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  (iii) a family of chain maps

$$\alpha = \left\{\alpha_n : \left(C_*\left(K_n\right), \partial\right) \to \left(Hom\left(A^{\otimes n}, A\right), \delta\right)\right\},$$

where α_n () $= m_n$ and $\delta\left(f\right) = d \circ f - (-1)^{|f|} f \circ d^{\otimes}$

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   $:=$  $($  $\otimes \mathbf{1})$ and  $:=$  $(\mathbf{1} \otimes$  $)$

  Then $\alpha($  $) = m_3$,

  $\alpha($  $) = m_2(m_2 \otimes \mathbf{1})$ and $\alpha($  $) = m_2(\mathbf{1} \otimes m_2)$

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- *m_3 is a chain homotopy from $m_2(m_2 \otimes \mathbf{1})$ to $m_2(\mathbf{1} \otimes m_2)$* called the **associator**

Tensor Product of A_∞-Algebras

- Given A_∞-algebras (A,α) and (B,β) and a diagonal $\Delta_K : C_*(K) \to C_*(K) \otimes C_*(K)$, the composition

$$\begin{array}{ccc} C_*(K_n) & \overset{\gamma}{\dashrightarrow} & Hom\left((A\otimes B)^{\otimes n}, A\otimes B\right) \\ \Delta_K \downarrow & & \uparrow \approx \\ C_*(K_n)\otimes C_*(K_n) & \underset{\alpha\otimes\beta}{\longrightarrow} & Hom(A^{\otimes n},A)\otimes Hom(B^{\otimes n},B) \end{array}$$

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- Δ_K *is the essential ingredient in the tensor product*

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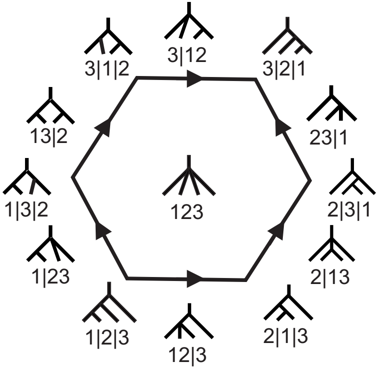
- Our proof that $\Delta_K = \Delta'_K$ views the permutahedron P_n as a subdivision of K_{n+1} and appeals to the combinatorics of P_n

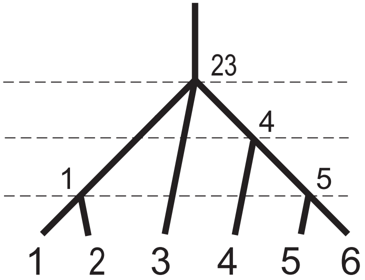
The Permutahedron P_n

P_n is an $(n-1)$-dimensional polytope constructed by P. Schoute (1911) whose faces are indexed either by planar rooted *leveled* trees (PLTs) with $n+1$ leaves or ordered partitions of $\underline{n} := \{1, 2, \ldots, n\}$



The Permutahedron P_3

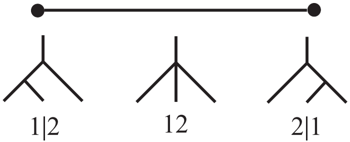
Partitions and PLTs



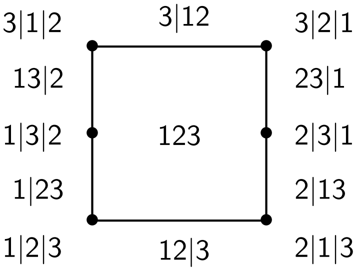
15|4|23

Inductive Construction of P_n

P_1



$P_2 = P_1 \times \mathtt{I}$



P_3 as a subdivision of $P_2 \times \mathtt{I}$

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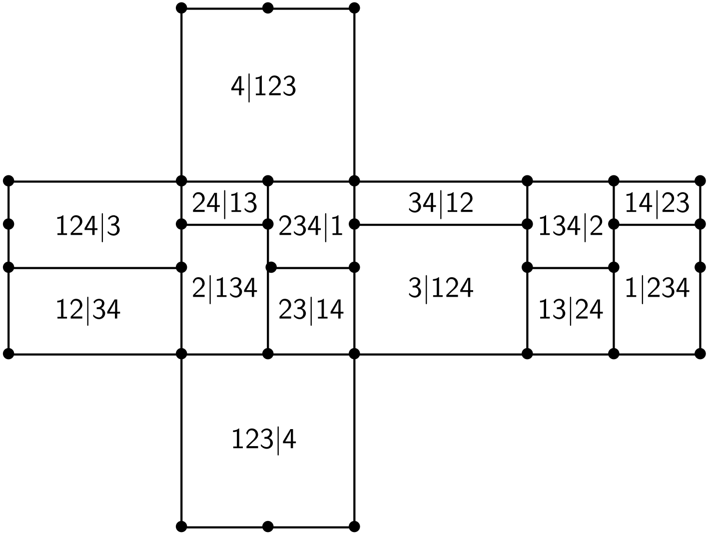

P_4 as a Subdivision of $P_3 \times \mathrm{I}$

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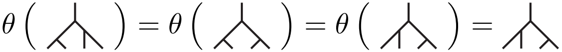

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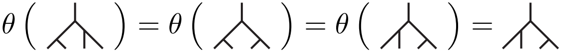
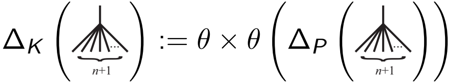


- A diagonal Δ_P on P_n induces a diagonal Δ_K on K_{n+1} :



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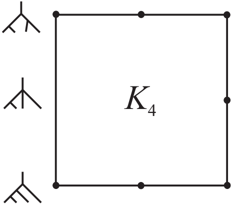

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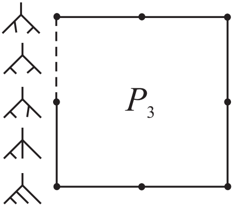

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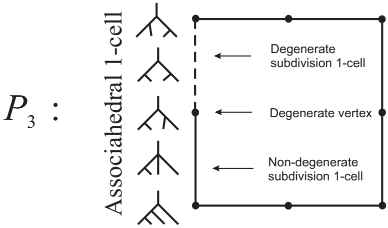

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- **Tamari order** is the restriction of weak order to vertices

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- For example, the SCP associated with $\sigma = 4|2|1|3$ is

$$a_\sigma \times b_\sigma = \overleftarrow{\sigma}_1|\overleftarrow{\sigma}_2 \times \overrightarrow{\sigma}_3|\overrightarrow{\sigma}_2|\overrightarrow{\sigma}_1 = 421|3 \times 4|2|13$$

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Step Matrix Representation

Given an SCP $A_1|\cdots|A_p \times B_q|\cdots|B_1$, construct the corresponding $q \times p$ **step matrix representation** :

Step matrix representation of $38|24|5|17|6 \times 8|34|257|16$

			1	6
	2	5	7	
3	4			
8				

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Derived matrix with $M_1 = M_2 = \{8\}, N_2 = \{5, 7\}$

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	2	5	7	
3	4			
8				

$\longrightarrow$

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	2			
3	4	5	7	
		8		

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Complementary Pairs

Given a $q \times p$ derived matrix with columns $A'_1 \ \cdots \ A'_p$ and rows $B'_1 \ \cdots \ B'_q$, the corresponding **complementary pair** (CP) is

$$A'_1|\cdots|A'_p \times B'_q|\cdots|B'_1$$

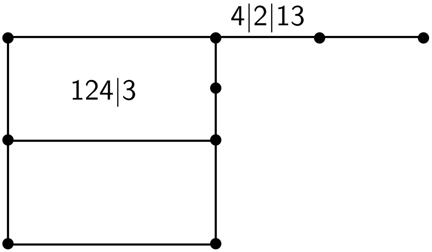
			1	6
	2			
3	4	5	7	
		8		

$\leftrightarrow$ 3|24|58|17|6 $\times$ 8|3457|2|16

Effect of Shift Actions on $P_n \times P_n$

1	3
2	
4	

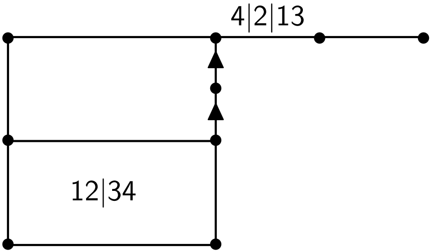
$124|3 \times 4|2|13 \quad \longleftrightarrow$



Right-Shifts Decrease Order of Left Factors

1	3
2	
	4

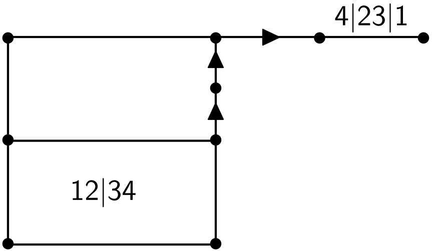
$12|34 \times 4|2|13 \qquad \longleftrightarrow$



Down-Shifts Increase Order of Right Factors

1	
2	3
	4

$12|34 \times 4|23|1 \quad \longleftrightarrow$



The S-U Diagonal Δ_P

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$\Delta_P(\underline{3})$ is the union of

$$\begin{aligned}
A_{1|2|3} \times B_{1|2|3} &= \{1|2|3 \times 123\} \\
A_{1|3|2} \times B_{1|3|2} &= \{1|23 \times 13|2\} \\
A_{2|1|3} \times B_{2|1|3} &= \{12|3 \times 2|13,\ 12|3 \times 23|1\} \\
A_{2|3|1} \times B_{2|3|1} &= \{2|13 \times 23|1\} \\
A_{3|1|2} \times B_{3|1|2} &= \{13|2 \times 3|12,\ 1|23 \times 3|12\} \\
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\end{aligned}$$

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A_{3|2|1} \times B_{3|2|1} &= \{123 \times 3|2|1\}
\end{aligned}$$

Extend Δ_P to faces of P_n multiplicatively :

$$\Delta_P(A_1|\cdots|A_p) = \Delta_P(A_1)|\cdots|\Delta_P(A_p)$$

The S-U Diagonal Δ_K

- A partition $A_1|\cdots|A_p$ is **degenerate** if for some j and k, there exist $x, z \in A_j$ and $y \in A_{j+k}$ such that $x < y < z$

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A_{2|3|1} \times B_{2|3|1} &= \{2|13 \times 23|1\} \\
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The Magical Formula

- A pair of faces $a \times b \subseteq K_{n+1} \times K_{n+1}$ is a **Matching Pair** (MP) if

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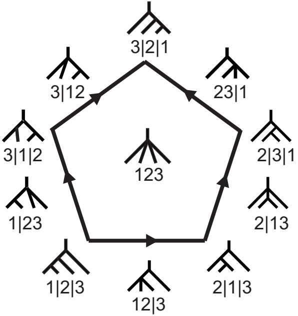
- The "Magical Formula" :

$$\Delta'_K\left(e^{n-1}\right) = \bigcup_{\substack{\text{MPs of faces} \\ a \times b \ \subseteq \ K_{n+1} \times K_{n+1}}} \{a \times b\}$$

Δ'_K on K_4 :



MPs $a \times b \subset K_4 \times K_4$:

$a \times b$	max a	min b
1|2|3 $\times$ 123	1|2|3	1|2|3
12|3 $\times$ 2|13	2|1|3	2|1|3
12|3 $\times$ 23|1	2|1|3	2|3|1
2|13 $\times$ 23|1	2|3|1	2|3|1
1|23 $\times$ 3|12	1|3|2	3|1|2
123 $\times$ 3|2|1	3|2|1	3|2|1

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- θ sends every non-degenerate CP to an MP by definition
- **Theorem** (S-U 2022) *Every MP is the image of a unique non-degenerate CP under θ; thus $\Delta'_K = \Delta_K$*

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- *Factors of non-degenerate CPs are minimal subdivision cells*

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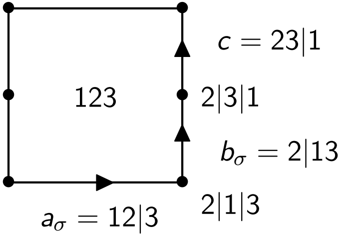
Example Consider $\sigma = 2|1|3$, the associated SCP $a_\sigma \times b_\sigma = 12|3 \times 2|13$ and the non-degenerate 1-cell $c = 23|1$. Then $\min b_\sigma = 2|1|3 < \min c = 2|3|1$ and c is the right factor of the CP $12|3 \times 23|1$ derived from $a_\sigma \times b_\sigma$.



Outline of Proof

- Identify each k-face $F \subseteq K_{n+1}$ with the corresponding associahedral k-cell $\mathcal{F} \subseteq P_n$; then $\theta(\mathcal{F}_{\min}) = F$

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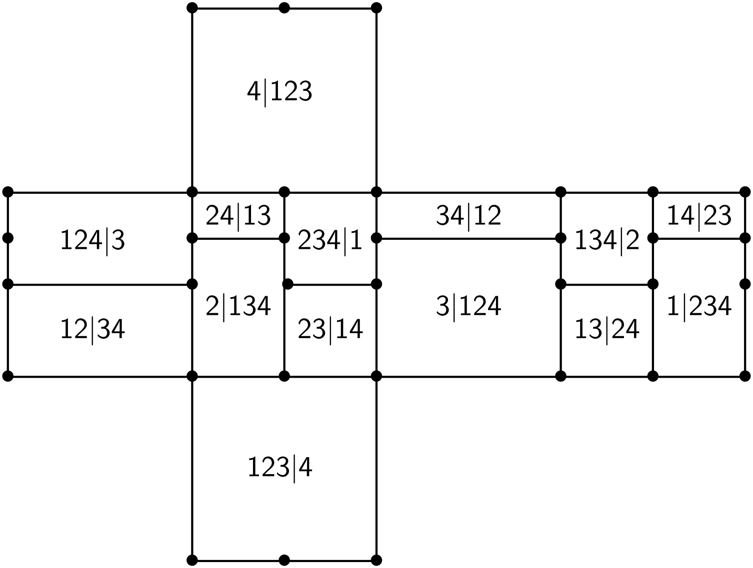
- Label F with the partition of $\mathcal{F}_{\text{min}}$

Outline of Proof



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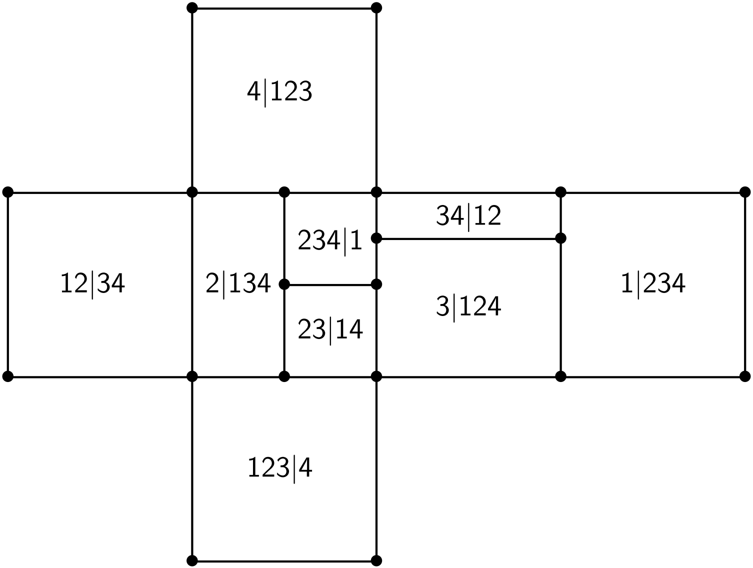

Outline of Proof

Theorem *If* $F \times G \subset K_{n+1} \times K_{n+1}$ *is an MP, then* $\mathcal{F}_{\min} \times \mathcal{G}_{\min} \subset P_n \times P_n$ *is a CP and* $\theta(\mathcal{F}_{\min}) \times \theta(\mathcal{G}_{\min}) = F \times G$*. Thus* $\Delta'_K = \Delta_K$.

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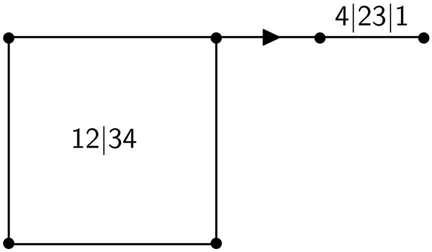
Example Consider the MP $F \times G = 12|34 \times 4|23|1 \subset K_5 \times K_5$



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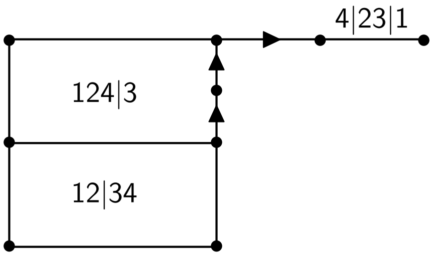

On P_4, $\mathcal{F} = 12|34 \cup 124|3$, $\mathcal{F}_{\min} \times \mathcal{G}_{\min} = 12|34 \times 4|23|1$ is a CP, and $\theta(12|43) \times \theta(4|23|1) = 12|34 \times 4|23|1$

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- If F is a k-face of K_{n+1}, $\mathcal{F}_{\text{min}}$ is the unqiue non-degenerate subdivision k-cell of the associahedral k-cell $\mathcal{F} \subset P_n$

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- If F is a k-face of K_{n+1}, $\mathcal{F}_{\text{min}}$ is the unqiue non-degenerate subdivision k-cell of the associahedral k-cell $\mathcal{F} \subset P_n$
- If $F \times G \subset K_{n+1} \times K_{n+1}$ is an MP, then $\mathcal{F}_{\text{min}} \times \mathcal{G}_{\text{min}}$ is a CP
- $\theta(\mathcal{F}_{\text{min}}) \times \theta(\mathcal{G}_{\text{min}}) = F \times G$

The End

THANK YOU!