Function-encoding Quantum States

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Tetrahedral Geometry-Topology Seminar
7 February 2020

Outline

1. Example

2. Motivation to Study Quantum Hypergraph States

3. Some results

4. Conclusion

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2. Motivation to Study Quantum Hypergraph States
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Ingredient 1: a Boolean function

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- $f(x, y, z) = x + xy + yz + xyz$

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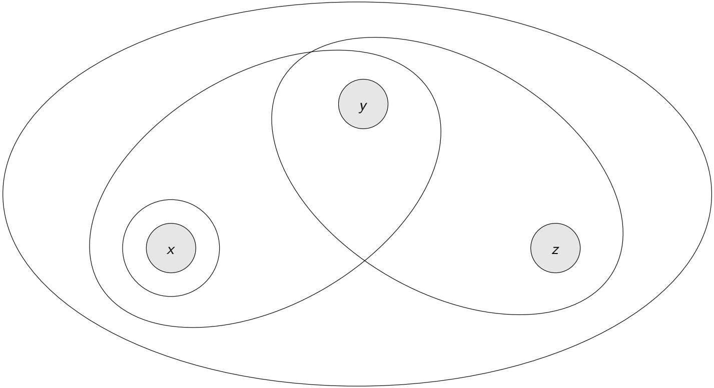
Just count! $\#\{f \colon B^n \to B\} = 2^{2^n}$

Ingredient 2: a hypergraph



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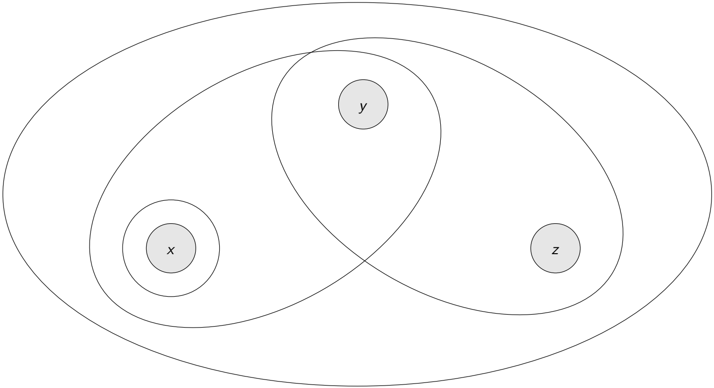

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(counting again) $\#\{\text{hypergraphs on } n \text{ vertices}\} = 2^{2^n}$

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- the space of 3-qubit states is $\mathbb{C}^8$
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$$|\psi_f\rangle := |000\rangle + |001\rangle + |010\rangle - |011\rangle - |100\rangle - |101\rangle + |110\rangle + |111\rangle$$

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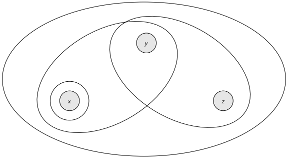
$[x + xy + yz + xyz]$ $\longleftrightarrow$  $\longleftrightarrow$ $|\psi_f\rangle$

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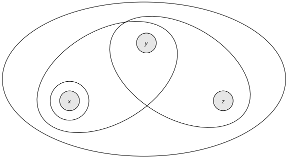
$[x + xy + yz + xyz]$ $\longleftrightarrow$  $\longleftrightarrow$ $|\psi_f\rangle$

- f: a polynomial in variables $x_1, x_2, \dots, x_n$
- G_f: a vertex for each variable, a (hyper)edge for each monomial in f
- $|\psi_f\rangle = \sum_{i_1, i_2, \dots, i_n} (-1)^{f(i_1, i_2, \dots, i_n)} |i_1 i_2 \dots i_n\rangle$

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Consider two specifications of a boolean function:

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Problem: how to convert between these two representations?

Conversion example: $|000\rangle + |001\rangle + |010\rangle - |011\rangle - |100\rangle - |101\rangle + |110\rangle + |111\rangle$ to $x + xy + yz + xyz$

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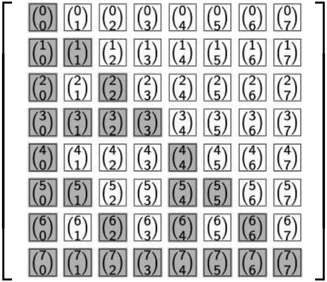

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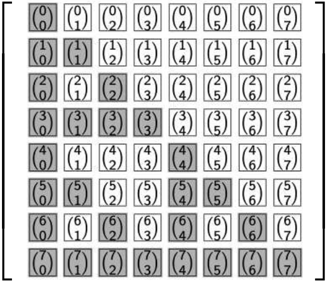

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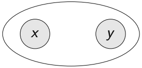
- foundational questions about nonlocality and entanglement properties
- "spooky action at a distance", the poster child of motivational stories

The EPR state

- polynomial xy
- hypergraph (graph, actually) 

- quantum state $|00\rangle + |01\rangle + |10\rangle - |11\rangle$

The EPR state, cont'd

Standard basis for 1-qubit state space $\mathbb{C}^2$

$$\bigcirc = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \bullet = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

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Another basis for $\mathbb{C}^2$

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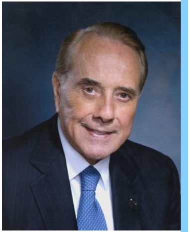
$$\text{EPR state} = \bigcirc\textcolor{red}{\bullet} + \bullet\textcolor{green}{\bullet}$$

Communication Task Scenario

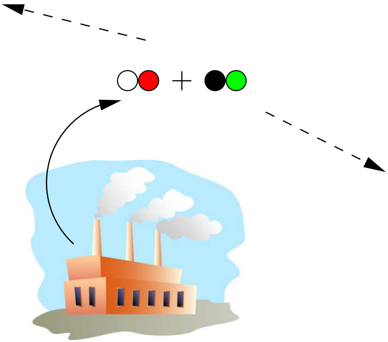
Alice and Bob run independent labs
may be separated by large distance

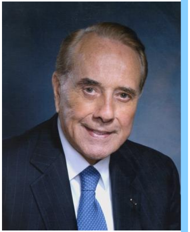


EPR Paradox

EPR Protocol Step 1

Factory prepares EPR state
Sends 1 qubit to Alice, 1 to Bob



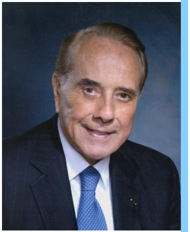


EPR Protocol Step 2

Alice measures his qubit

I got ●

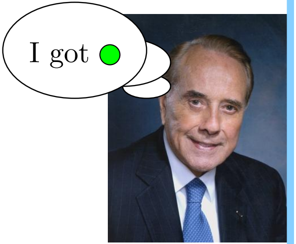
Post measurement state is ●●



EPR Protocol Step 3

Bob measures his qubit

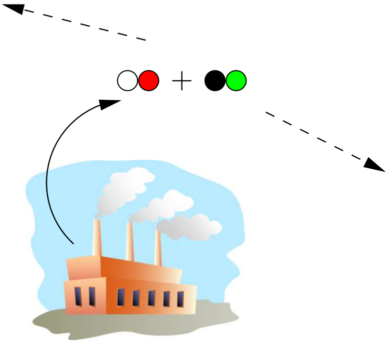
Post measurement state is ●●

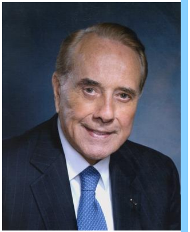


EPR Protocol Step 1, again

Factory prepares EPR state

Sends 1 qubit to Alice, 1 to Bob





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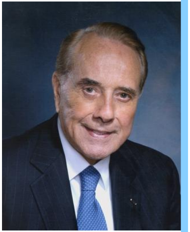
Post measurement state is ○●



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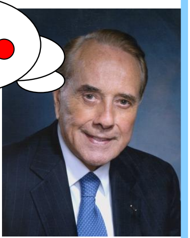

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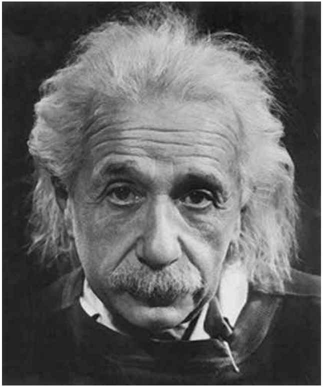
The first experiment to demonstrate the EPR measurement was by Alain Aspect in 1982.

Intellectual dissonance



"... spooky action at a distance"

Outline

1 Example

2 Motivation to Study Quantum Hypergraph States

3 Some results

4 Conclusion

Local Unitary (LU) Action

A 2×2 unitary acts on a vector in $\mathbb{C}^2$

$$U = \begin{bmatrix} a & b \\ c & c \end{bmatrix}$$

$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

$$U\,|\psi\rangle = (a\alpha + b\beta)\,|0\rangle + (c\alpha + d\beta)\,|1\rangle$$

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An n-tuple of 2×2 unitaries acts on a vector in $(\mathbb{C}^2)^{\otimes n}$

$$(U_1, U_2, \ldots, U_n) |\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots \otimes |\psi_n\rangle$$
$$= (U_1 |\psi_1\rangle) \otimes (U_2 |\psi_2\rangle) \otimes \cdots \otimes (U_n |\psi_n\rangle)$$

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$$\begin{aligned} Z_1 Z_2(\text{EPR}) &= (Z \otimes Z)(|00\rangle + |01\rangle + |10\rangle - |11\rangle) \\ &= |00\rangle - |01\rangle - |10\rangle - |11\rangle \end{aligned}$$

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A particular state may have more subtle LU stabilizers.

$$\begin{aligned} &[\exp(itZ) \otimes \exp(-itX)]\,(\mathrm{EPR}) \\ &= \left(\begin{bmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{bmatrix} \otimes \begin{bmatrix} \cos t & -i \sin t \\ -i \sin t & \cos t \end{bmatrix} \right) (\mathrm{EPR}) \\ &= \mathrm{EPR} \end{aligned}$$

LU equivalence

Entanglement

LU-equivalent states share the same nonlocal properties. Such states are said to have the same *entanglement type*.

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Central question

Classify the possible entanglement types, that is, classify orbits of the LU action on state space.

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Under any group action on any set, if two points are equivalent under the group action, then the stabilizer subgroups of those points are isomorphic.

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Consequence

LU stabilizers are entanglement invariants.

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Motivations: codes, nonlocality

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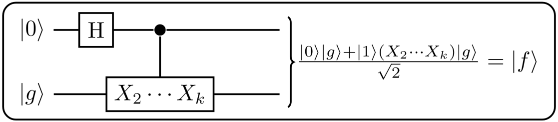
Construction of $X^{\otimes k}$-stable states"



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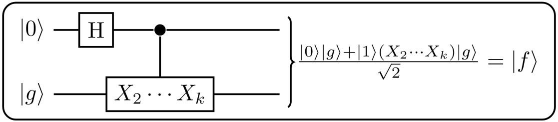

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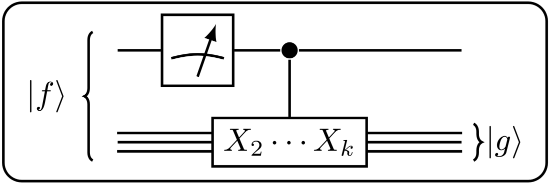

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- polynomials are in n variables over $\mathbb{F}_d$
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- d-hypergraphs seem to be of limited usefulness
- generalizations of Pauli X, Z

Sample result

Theorem (classification of local generalized Pauli classes for $d = 3$, $n = 2$)

- 0 (this is the class of the product state)
- xy
- xy^2
- x^2y
- $x^2y + y^2x$
- x^2y^2
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There are $d^{(d^n)}$ functions $(\mathbb{F}_d)^n \to \mathbb{F}_d$, so $3^{(3^2)} = 19,683$ for $d = 3, n = 2$

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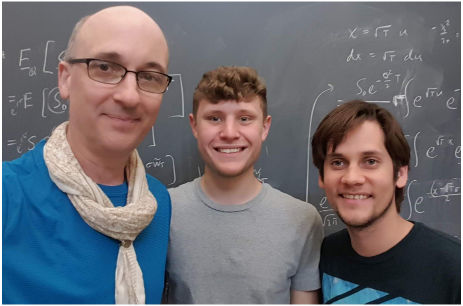
Work on hypergraph states is. . .

- well motivated
- full combinatorial elegance and fun
- cool results
- short on general results
- still in search of a killer app

Thank you!

LVC Mathematical Physics Research Group



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http://quantum.lvc.edu/mathphys