Tetrahedral Geometry and Topology Seminar

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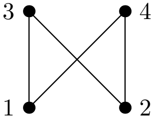
October 14, 2016

Seaweed and poset algebras - synergies and cohomolgoy

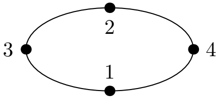


Part II
Cohomology of Lie Poset Algebras

by
Vincent E. Coll, Jr.

A report on joint work
with
Murray Gerstenhaber**

**University of Pennsylvania



A Nexus for Cohomology Theories

Lie algebra, $\mathfrak{g}$
Chevalley-Eilenberg

Poset algebras

Associative algebra, A
Hochschild

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Simplicial, Σ
deRham

Hochschild

$$(\delta F^n)(a_1,\dots,a_{n+1}) = a_1F(a_2,\dots,a_{n+1}) + \sum_{i=1}^{n}(-1)^iF(\dots,a_ia_{i+1},\dots) + (-1)^{n+1}F(a_1,\dots,a_n)a_{n+1}$$

Chevalley-Eilenberg

$$(\delta F^n)(g_1,\dots,g_{n+1}) = \sum_{1\le i\le n}(-1)^i[g_i,F(g_1\dots,\hat{g}_i,\dots g_{n+1})]+ \sum_{1\le i<j\le n}(-1)^{i+j}F([g_i,g_j],g_1\dots,\hat{g}_i,\dots,\hat{g}_j,\dots g_{n+1})$$

The Complexes

Hochschild

$$
\begin{aligned}
(\delta F^n)(a_1, \dots, a_{n+1}) &= a_1 F(a_2, \dots, a_{n+1}) + \sum_{i=1}^{n} (-1)^i F(\dots, a_i a_{i+1}, \dots) \\
&\quad + (-1)^{n+1} F(a_1, \dots, a_n) a_{n+1}
\end{aligned}
$$

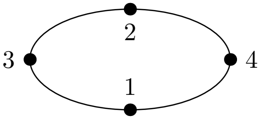
Chevalley-Eilenberg

$$
\begin{aligned}
(\delta F^n)(g_1, \dots, g_{n+1}) &= \sum_{1 \le i \le n} (-1)^i [g_i, F(g_1 \dots, \hat{g}_i, \dots g_{n+1})] + \\
&\sum_{1 \le i < j \le n} (-1)^{i+j} F([g_i, g_j], g_1 \dots, \hat{g}_i, \dots, \hat{g}_j, \dots g_{n+1})
\end{aligned}
$$

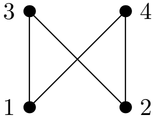
Cases of special interest

$H^*(A, A)$ and $H^*(\mathfrak{g}, \mathfrak{g})$

The Construction



Σ



$\mathcal{P}(\Sigma)$

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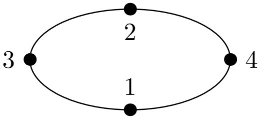
$A(\mathcal{P})$ and $\mathfrak{g}(\mathcal{P})$

Poset Algebras

The Construction



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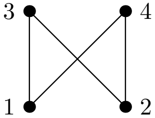

$\mathcal{P}(\Sigma)$

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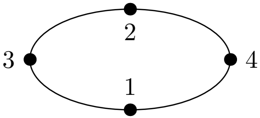
$A(\mathcal{P})$ and $\mathfrak{g}(\mathcal{P})$

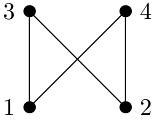
Theorem (Gerstenhaber and Schack, '83)

$$H^*(\Sigma, \mathbf{k}) \cong H^*(A(\mathcal{P}), A(\mathcal{P})).$$

Poset Algebras

The Construction





$$\begin{pmatrix} * & 0 & * & * \\ 0 & * & * & * \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{pmatrix}$$

Σ $\qquad$ $\mathcal{P}(\Sigma)$ $\qquad$ $A(\mathcal{P})$ and $\mathfrak{g}(\mathcal{P})$

Theorem (Gerstenhaber and Schack, '83)

$$H^*(\Sigma, \mathbf{k}) \cong H^*(A(\mathcal{P}), A(\mathcal{P})).$$

Application

$$H^2(A(S^1), A(S^1)) = 0$$

Theorem (Gestenhaber and Schack, '86)

Hochschild Cohomology can be computed relative to a separable subalgebra.

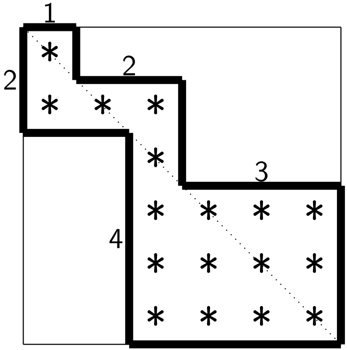
R-relative cohomology

Axiom 1 Elements of R can "float"

i. $F(\ldots, a_i r, a_{i+1}, \ldots) = F(\ldots, a_i, r a_{i+1}, \ldots)$,
ii. $F(a_1, \ldots, a_n r) = F(a_1, \ldots, a_n) r$,
iii. $F(r a_1, \ldots, a_n) = r F(a_1, \ldots, a_n)$ for $a_i \in A, r \in R$

Axiom 2 $F(a_1, \ldots, a_n) = 0$, if any $a_i \in R$.

Seaweeds...



...are cohomologically dead!

What about the Lie poset case?

In what setting is $H^*(\mathfrak{g}(\mathcal{P}), \mathfrak{g}(\mathcal{P}))$ tractable?

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Toral Actions

$\sigma \in \mathfrak{g}$

Action on $\mathfrak{g}$: $[\sigma, -]$

(Compatible) Action on M: $\sigma_1\sigma_2 m - \sigma_2\sigma_1 m = [\sigma_1, \sigma_2]m$

Definition

Let σ be toral and F^n be an n-cocycle

$[\sigma, a_i] = w_i a_i, i = 1, ..., n$

F^n is homogeneous of weight r if $F^n(a_1, ..., a_n)$ is homogeneous of weight $\sum w_i + r$

N.B. Can extend to a commuting family acting torally

The descent to cohomology-Vivianni's Lemma

Theorem (Gerstenhaber, '12; Vivianni, '09)

An n-cocycle F is cohomologous to its homogeneous weight 0 part.

$$H^*(\mathfrak{g}, M) \cong H^*(\mathfrak{g}, M)_0.$$

Question: Where do we see such actions?

Answer

$$\mathfrak{g} = \mathfrak{h} \ltimes \mathfrak{k},$$

where $\mathfrak{h}$ is abelian and acts torally both on $\mathfrak{g}$ and on a $\mathfrak{g}$-module M

Theorem (C. and Gerstenhaber, '16)

With respect to the weighting induced by the action of $\mathfrak{h}$ we have

$$H^*(\mathfrak{g}, M) \cong \bigwedge \mathfrak{h}^\vee \bigotimes H^*(\mathfrak{k}, M)_{\mathbf{0}}.$$

In what setting is this formula tractable?

Answer

$\mathfrak{sl}(N, \mathbf{k}) = N \times N$ matrices over $\mathbf{k}$ of trace 0.

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Theorem (C. and Gerstenhaber, '16)

If $\mathfrak{h} \subset \mathfrak{g} \subset \mathfrak{b}$ then $\mathfrak{g}$ must be of the form $\mathfrak{g}(\mathcal{P})$.

Lie Poset Subalgebras of $\mathfrak{sl}(N, \mathbf{k})$

Answer

$\mathfrak{sl}(N, \mathbf{k}) = N \times N$ matrices over $\mathbf{k}$ of trace 0.

Theorem (C. and Gerstenhaber, '16)

If $\mathfrak{h} \subset \mathfrak{g} \subset \mathfrak{b}$ then $\mathfrak{g}$ must be of the form $\mathfrak{g}(\mathcal{P})$.

Theorem (C. and Gerstenhaber, '16)

With the foregoing hypotheses,

$$H^*(\mathfrak{g}(\mathcal{P}), \mathfrak{g}(\mathcal{P})) = \bigwedge \mathfrak{h}^\vee \bigotimes H^*(\Sigma^+, \mathbf{k}).$$

Poset Algebra Cohomology - $\mathfrak{sl}(N, \mathbf{k})$

Associative case: $H^*(A(\mathcal{P}), A(\mathcal{P})) \cong H^*(\Sigma, \mathbf{k}).$

Lie case: $H^*(\mathfrak{g}(\mathcal{P}), \mathfrak{g}(\mathcal{P})) \cong \bigwedge \mathfrak{h}^\vee \bigotimes H^*(\Sigma^+, \mathbf{k}).$

Conjecture

True for other simple algebras in the great classification

One more time

$$\Sigma = S^1$$

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$$H^2(A(S^1), A(S^1)) = 0 \qquad\qquad H^2(\mathfrak{g}(S^1), \mathfrak{g}(S^1)) \neq 0$$

$$\mathfrak{g}(S^1) = \mathfrak{h} \ltimes \mathfrak{k}$$

Basis for $\mathfrak{h} = \{\eta_1, \eta_2, \eta_3\}$ Basis for $\mathfrak{k} = \{e_{13}, e_{14}, e_{23}, e_{24}\}$

A non-trivial deformation

$$[\eta_1, e_{12}]_t = (2 + t)e_{12}$$

Questions - and some answers

- **Question 1:** Characteristic p for $\mathfrak{sl}(N, \mathbf{k})$?
- **Question 2:** Other Chevalley type Lie algebras?

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- **Question 1:** Characteristic p for $\mathfrak{sl}(N, \mathbf{k})$?
- **Question 2:** Other Chevalley type Lie algebras?

$$H^2(\mathfrak{g}, \mathfrak{g}) = (\bigwedge^2 \mathfrak{h}^\vee \bigotimes \mathfrak{c}) \bigoplus (\mathfrak{h}^\vee \bigotimes H^1(\Sigma, \mathbf{k})) \bigoplus (H^2(\Sigma^+, \mathbf{k}))$$