Tetrahedral Geometry and Topology Seminar
Deformation Theory Seminar

October 14, 2016

Seaweed and poset algebras
synergies and cohomology

Part I
Seaweeds

by

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A report on joint work
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Objects and definitions

OBJECTS

Frobenius seaweed subalgebras of simple Lie algebras

Simple

$A_n = \mathfrak{sl}(n)$, $B_n = \mathfrak{so}(2n+1)$, $C_n = \mathfrak{sp}(2n)$, $D_n = \mathfrak{so}(2n)$

E_6, E_7, E_8, F_4, G_2

Frobenius

$$B_F[x,y] = F[x,y]$$

$$\text{ind } \mathfrak{g} = \min_{F \in \mathfrak{g}^*} \dim \ker B_F.$$

Seaweeds

$\mathfrak{g}$ simple

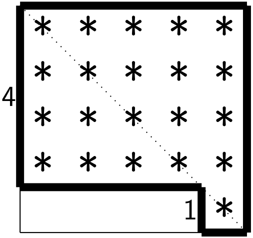
$\mathfrak{p}$, $\mathfrak{p}'$ parabolic subalgebras with $\mathfrak{p} + \mathfrak{p}' = \mathfrak{g}$

$\mathfrak{p} \cap \mathfrak{p}'$ is a seaweed

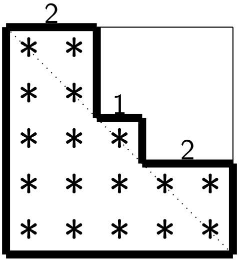
Special linear Lie algebra

$$\mathfrak{sl}(n) = \{A \in \text{gl}(n) \mid \text{tr}(A) = 0\}$$

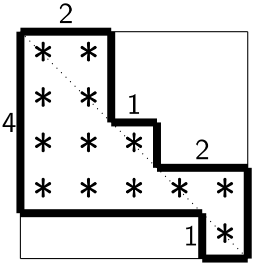
Type-A seaweed



$\mathfrak{p}$



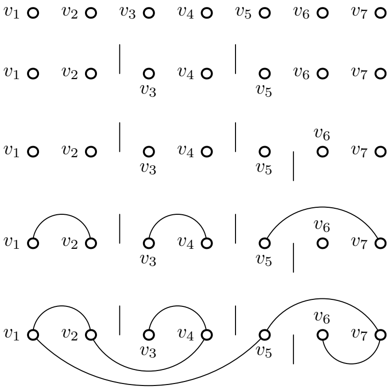
$\mathfrak{p}'$



$\mathfrak{p} \cap \mathfrak{p}' = \mathfrak{p}_5^{\text{A}} \frac{4|1}{2|1|2}$

Index computation – Meanders

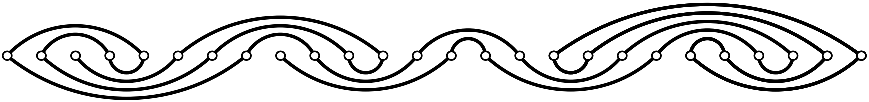
$$\text{Type } \mathfrak{p}_7^{\mathrm{A}} \frac{2|2|3}{5|2}$$



Theorem (Dergachev and Kirillov, 2000)

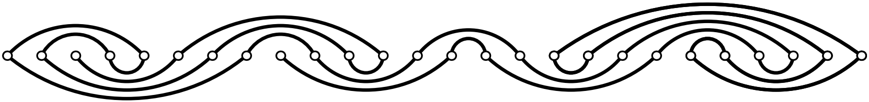
For a seaweed $\mathfrak{g} \subseteq \mathfrak{sl}(n)$, $\operatorname{ind} \mathfrak{g} = 2C + P - 1$

Not so easy in practice

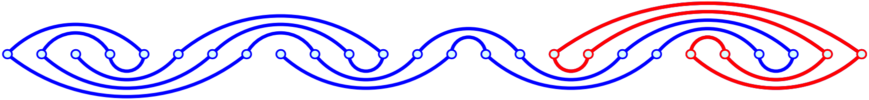


Type: $\frac{5|7|4|10}{8|6|6|6}$

Not so easy in practice



Type: $\frac{5|7|4|10}{8|6|6|6}$



Index: 2

Index formulas

Theorem (Elashvilli (1990), C., Magnant, and Giaquinto (2010))

If $n = a + b$ *with* $(a, b) = 1$, *then* $\frac{a|b}{n}$ *is Frobenius.*

Theorem (C., Hyatt, Magnant, and Wang (2015))

If $n = a + b + c$ *with* $(a + b, b + c) = 1$, *then* $\frac{a|b|c}{n}$ *is Frobenius.*

Theorem (Karnauhova and Liebscher (2015))

If $m \geq 4$, *then* $\nexists$ *homogeneous* $f_1, f_2 \in \mathbb{Z}[x_1, \ldots, x_m]$ *such that*

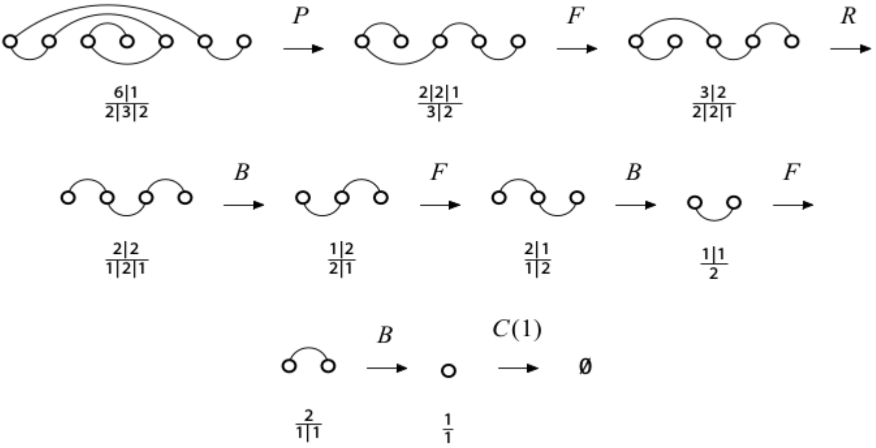
$$ind \frac{a_1|a_2| \cdots |a_m}{n} = \gcd(f_1(a_1, \ldots, a_m), f_2(a_1, \ldots, a_m)).$$

The signature of a meander

$$\frac{a_1|a_2|\cdots|a_m}{b_1|b_2|\cdots|b_t} \longrightarrow M$$

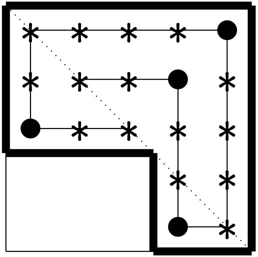
Condition	**Move**
$a_1 = b_1$	Component removal
$a_1 = 2b_1$	Block removal
$b_1 < a_1 < 2b_1$	Rotation
$a_1 > 2b_1$	Pure
$a_1 < b_1$	Flip

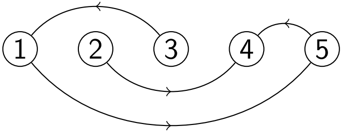
Detailed signature of $\mathfrak{p}_7^{\mathrm{A}}\frac{6|1}{2|3|2}$



Type-A Frobenius functional, F_A

$\mathfrak{p}_5^{\mathrm{A}}\,\frac{3|2}{5}$

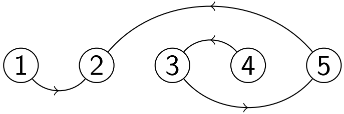




$F_A = e^*_{1,5} + e^*_{2,4} + e^*_{3,1} + e^*_{5,4}$ is regular

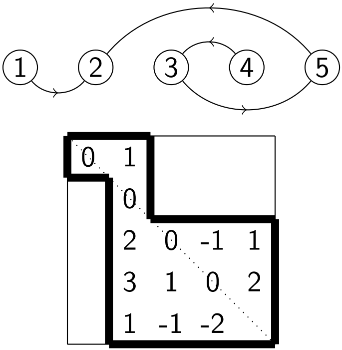
Type-A principal element, $\hat{F}_A$

$\mathfrak{p}_5^{\mathrm{A}}\,\frac{1|4}{2|3}$



- Pick an endpoint of the path, say vertex 4
- Follow the path from 1 to 4, counting the arrows (**measure**)
- The measure is -2, this is the first diagonal entry of D.
- Repeat for each vertex: $D = \mathrm{diag}(-2, -3, -1, 0, -2)$
- Normalize: $\hat{F}_A = D + 8/5 = \mathrm{diag}(-2/5, -7/5, 3/5, 8/5, -2/5)$

Eigenvalues of $\mathrm{ad}\hat{F}_A$



- Pick a pair of vertices (4,2)
- Measure is 3
- This is an eigenvalue: $\mathrm{ad}\hat{F}_A(e_{4,2}) = 3e_{4,2}$

Eigenvalues	-2	-1	0	1	2	3
Dimensions	1	2	4	4	2	1

Type-A unbroken spectrum result

Confirming a claim of Gerstenhaber and Giaquinto – and a bit more.

Theorem (C., Hyatt, Magnant (2016))

For a seaweed subalgebra of $\mathfrak{sl}(n)$, the spectrum of $\mathrm{ad}\hat{F}$ is an unbroken sequence of integers. Moreover, the multiplicities form a symmetric distribution.

Proof uses the Signature.

Symplectic Lie algebra

$$\mathfrak{sp}(2n) = \left\{ \begin{bmatrix} A & B \\ C & -\widehat{A} \end{bmatrix} : B = \widehat{B}, C = \widehat{C} \right\},$$

- A, B, and C are $n \times n$ matrices.
- $\widehat{A}$ is the transpose of A with respect to the anitdiagonal.

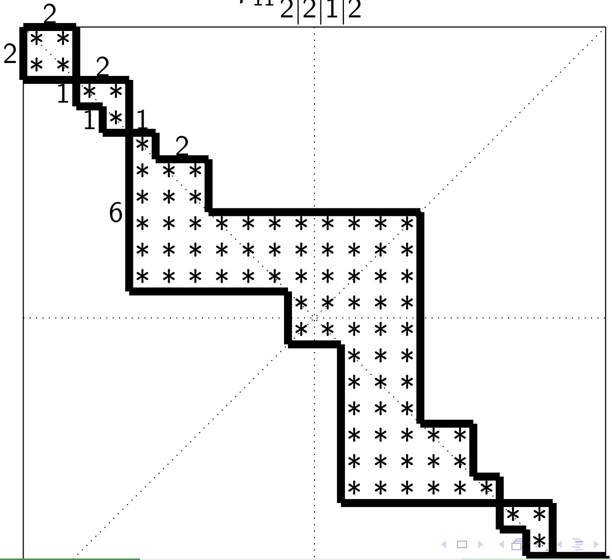
Type-C seaweeds

$$\mathfrak{p}_n^{\mathrm{C}} \frac{a_1 | \cdots | a_m}{b_1 | \cdots | b_t}$$

where the $a_1 | \cdots | a_m$ and $b_1 | \cdots | b_t$ are **partial compositions** of n, i.e.

$$\sum a_i \leq n \text{ and } \sum b_i \leq n.$$

$$\mathfrak{p}_{11}^{\mathrm{C}} \frac{2|1|1|6}{2|2|1|2}$$



...and the associated meander of

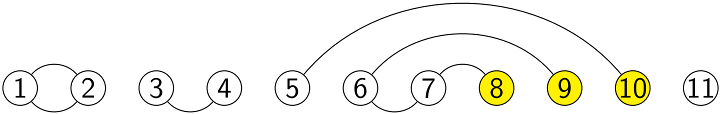
Blocks

$$V_i = \{vertices : a_1 + a_2 + \cdots + a_{i-1} < \; vertex \; label \; < a_1 + a_2 + \cdots a_i + 1$$

Tail

Let $r = n - \sum a_i$ and $T_n(\underline{a}) = \{n - r + 1, n - r + 2, \ldots, n\}$

$$T = (T_n(\underline{a}) \cup T_n(\underline{b})) \setminus (T_n(\underline{a}) \cap T_n(\underline{b}))$$



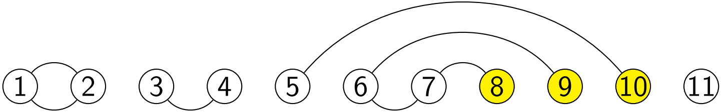
- Top blocks: $V_1 = \{1, 2\}$, $V_2 = \{3\}$, $V_3 = \{4\}$, $V_4 = \{5, 6, 7, 8, 9, 10\}$
- Bottom blocks: $V_1 = \{1, 2\}$, $V_2 = \{3, 4\}$, $V_3 = \{5\}$, $V_4 = \{6, 7\}$
- Tail: $T = \{8, 9, 10\}$

Index computation for $\mathfrak{p}_{11}^{\mathrm{C}} \frac{2|1|1|6}{2|2|1|2}$

Theorem (C., Hyatt., and Magnant (2016))

For a seaweed $\mathfrak{g} \subseteq \mathfrak{sp}(2n)$, $\mathrm{ind}\, \mathfrak{g} = 2C + \tilde{P}$

$\tilde{P}$ *is number of paths with 0 or 2 edges in the tail.*



component	# of tail vertices	cycle?	contribution to index
$G[\{1,2\}]$	0	yes	2
$G[\{3,4\}]$	0	no	1
$G[\{\{11\}]$	0	no	1
$G[\{6,7,8,9\}]$	2	no	1
$G[\{5,10\}]$	1	no	0

Index Formulas

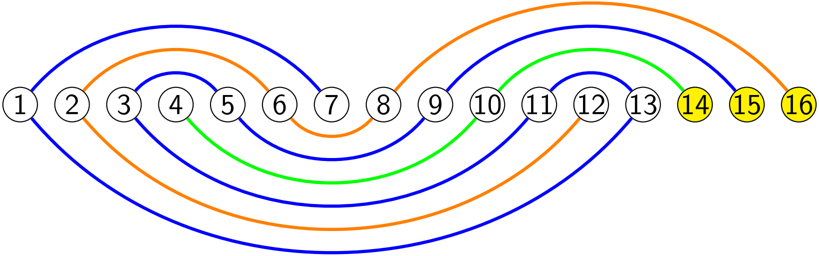
Theorem (C., Hyatt, and Magnant)

$\text{ind}\, \mathfrak{p}_n^{\mathrm{C}} \frac{n}{a|b} = 0$ *iff one of the following holds:*

- $a + b = n - 1$ *and* $\gcd(a + b, b + 1) = 1$
- $a + b = n - 2$ *and* $\gcd(a + b, b + 2) = 1$
- $a + b = n - 3$ *and* $\gcd(a + b, b + 3) = 2$ *with* $n, a,$ *and* b *all odd.*

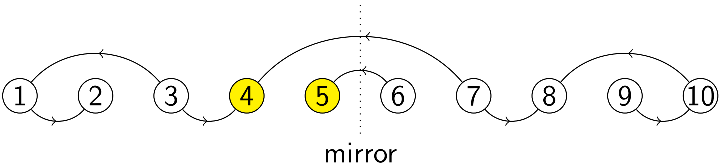
Corollary – Frobenius Type-C meanders are ...

...a certain kind of forest...



Type-C Frobenius functional, F_C

Panyushev-Yakimova meander



Theorem (C., Hyatt, and Magnant (2015))

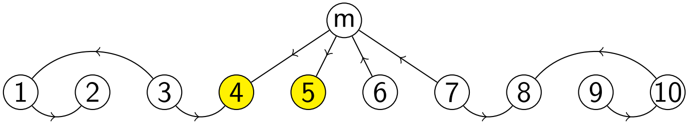
$$F_C = \sum_{(i,j)} e^*_{i,j}, \textit{ such that } i \le n \textit{ or } j \le n, \textit{ is Frobenius.}$$

Example

$F_C = e^*_{1,2} + e^*_{3,4} + e^*_{3,1} + e^*_{6,5} + e^*_{7,4}$ is Frobenius.

Type-C principal element, $\hat{F}_C$

Principal Graph

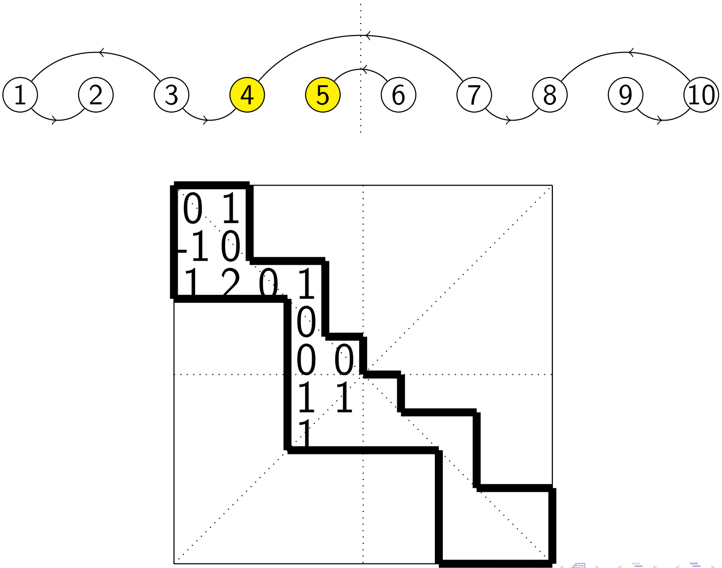


Edges incident with m have measure 1/2, all other edges have measure 1.

Measure from each vertex to m.

$$\hat{F}_C = \text{diag}\left(-\frac{1}{2}, -\frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{3}{2}, \frac{1}{2}\right)$$

Eigenvalues of $\hat{F}_C$



Type-C unbroken spectrum result

Eigenvalues	-1	0	1	2
Dimensions	1	6	6	1

Type-C unbroken spectrum result

Eigenvalues	-1	0	1	2
Dimensions	1	6	6	1

Theorem (C., Hyatt., and Magnant (2016))

The spectrum of a principal element of Frobenius symplectic seaweed is an unbroken sequence of integers. Moreover the multiplicities form a symmetric sequence.

Proof uses two ingredients

- Type-A unbroken result
- Adaptation of the signature

Special orthogonal Lie algebra

$$\mathfrak{so}(2n+1) = \{A \in \mathrm{gl}(n) \mid A = -\widehat{A}\},$$

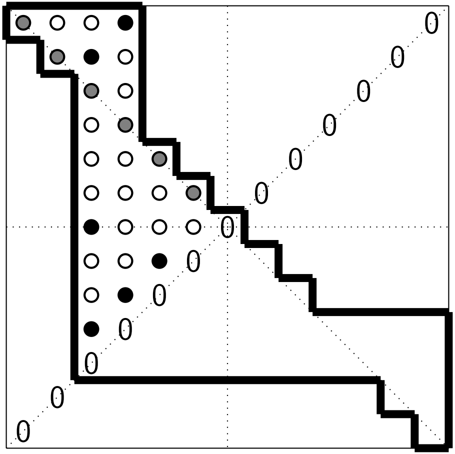
$$\mathfrak{p}_n^{\mathrm{B}} \frac{a_1|\cdots|a_m}{b_1|\cdots|b_t}$$

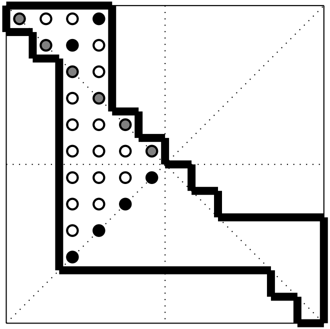
where the $a_1|\cdots|a_m$ and $b_1|\cdots|b_t$ are **partial compositions** of n, i.e.

$$\sum a_i \leq n \text{ and } \sum b_i \leq n.$$

Type-B seaweeds cont...

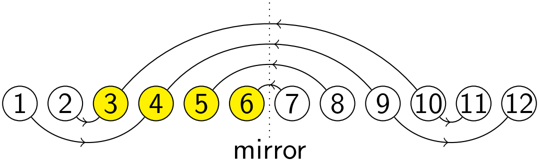
$$\mathfrak{p}_5^{\mathrm{B}}\,\frac{1|1}{4|1|1}$$





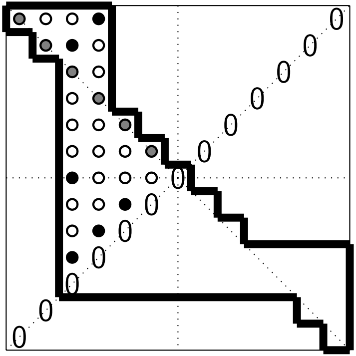
$$F_C = e^*_{1,4} + e^*_{2,3} + e^*_{10,3} + e^*_{9,4} + e^*_{8,5} + e^*_{7,6}$$

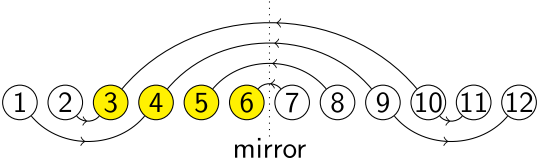
Read DIRECTLY from the meander



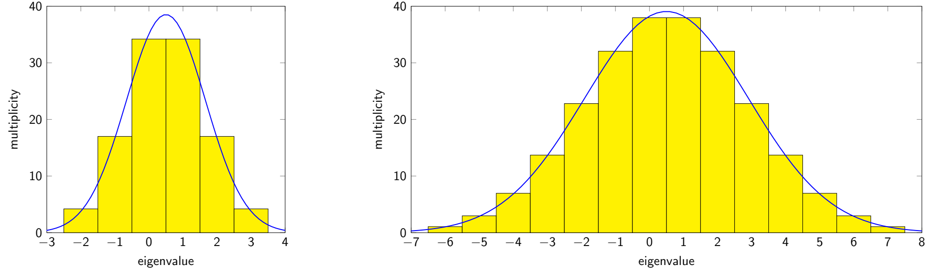
$\mathfrak{p}_5^{\mathrm{B}}\frac{1|1}{4|1|1}$

$$F_B = e^*_{1,4} + e^*_{2,3} + e^*_{10,3} + e^*_{9,4} + e^*_{8,5} + e^*_{7,3}$$





(It seems that) A stochasitic process is present



Distribution of eigenvalues of $\mathfrak{p}^{\mathrm{A}}_{17}\dfrac{4|10|3}{6|4|7}$ (left) and $\mathfrak{p}^{\mathrm{A}}_{17}\dfrac{4|13}{17}$ (right).

More precisely...

What seems to be true

1. Distribution is unimodular
2. Let $\lambda \in \mathbb{Z}^+$

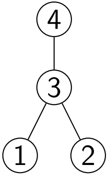
F_λ be a Frobenius seaweed with spectrum $\{1-\lambda, ..., 0, ..., \lambda\}$
$d_i(F_\lambda) =$ dim of the λ-eigenspace
$\{F_\lambda\}_{\lambda=1}^{\lambda=\infty}$ be a sequence of such Frobenius seaweeds

$\{X_\lambda\}_{\lambda=1}^{\lambda=\infty}$ be a sequence of random variables
$$P(X_\lambda = i) = \frac{d_i(F_\lambda)}{dimF_\lambda}$$

$X_\lambda \longrightarrow$ Normal (in distribution).

Lie poset algebras

The "Stargate" poset



$$\mathcal{P} = \{1, 2, 3, 4\}$$
$$1, 2 \preceq 3 \preceq 4$$

$$\begin{pmatrix} * & 0 & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix}$$

$$\text{Rank } \mathfrak{g}(\mathcal{P}, \mathbb{C}) = 3$$
$$\dim \mathfrak{g}(\mathcal{P}, \mathbb{C}) = 8$$

$$F = e^*_{1,4} + e^*_{2,4} + e^*_{2,3}$$
$$\hat{F} = \mathrm{diag}\left(\tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}\right)$$
$$\text{Spec ad}\hat{F} = \{0, 0, 0, 0, 1, 1, 1, 1\}.$$

But,...

$\mathfrak{g}(\mathcal{P}, \mathbb{C})$ is NOT a seaweed!

The only seaweed subalgebra of $\mathfrak{sl}(4)$ with Rank 3 and dimension 8 is

$$\mathfrak{p}_4^{\mathrm{A}} \frac{2|2}{1|3}$$

It's spectrum is $\{-1, 0, 0, 0, 1, 1, 1, 2\}$

But,...

$\mathfrak{g}(\mathcal{P}, \mathbb{C})$ is NOT a seaweed!

The only seaweed subalgebra of $\mathfrak{sl}(4)$ with Rank 3 and dimension 8 is

$$\mathfrak{p}_4^{\mathrm{A}} \frac{2|2}{1|3}$$

It's spectrum is $\{-1, 0, 0, 0, 1, 1, 1, 2\}$

Question: What is the larger category?