Discrete symmetries of hypergraph states

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Lebanon Valley College

Tetrahedral Geometry-Topology Seminar
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Outline

1. Basics
2. Graphs and Graph States
3. Hypergraphs and Hypergraph States
4. Symmetry, Geometry, and Combinatorics
5. Summary and Looking Forward

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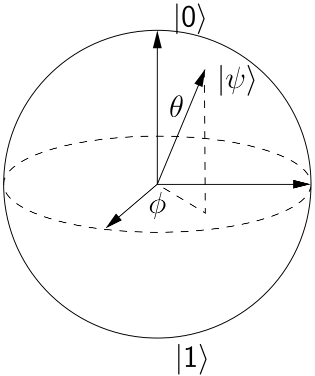
- we speak loosely and write the vector $\alpha |0\rangle + \beta |1\rangle$ but always mean its equivalence class in $\mathbb{P}^1$

The Bloch Sphere

$$S^2 \longleftrightarrow \mathbb{C}^2$$

$$(\theta, \phi) \longleftrightarrow \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$$



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- states are points in projective space
- write $|011\rangle$ for $|0\rangle \otimes |1\rangle \otimes |1\rangle$
- standard (computational) basis vectors have form

$$|I\rangle = |i_1 i_2 \ldots i_n\rangle\,, \qquad i_k = 0, 1, \qquad 1 \leq k \leq n$$

Entanglement

Entangled States

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Example: $|00\rangle + |11\rangle \neq (a|0\rangle + b|1\rangle) \otimes (c|0\rangle + d|1\rangle)$ for any a, b, c, d
Proof: Just look at

$$(a|0\rangle + b|1\rangle) \otimes (c|0\rangle + d|1\rangle) = ac|00\rangle + ad|01\rangle + bc|10\rangle + bd|11\rangle .$$

Terms don't work out.

Nonlocality

Spooky action at a distance

Alice has qubit 1 and Bob has qubit 2 of state $|00\rangle + |11\rangle$ in labs separated far apart. Each measures 0 or 1 with probability $1/2$, but they obtain the same outcome (both 0 or both 1) with probability 1.

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Motivation to study multiqubit states

Multiqubit states encode *data* and can be *processed* to perform algorithms and secure communication in ways that are (believed to be) not achievable with classical processing of classical bits. Entanglement and nonlocality play a role of essential resources for the speed up over classical algorithms.

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$$|+\rangle = |0\rangle + |1\rangle$$

Observation: $|+\rangle^{\otimes n} = \sum_I |I\rangle$

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The 2-qubit C operator (controlled-Z)

$$a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle \to a|00\rangle + b|01\rangle + c|10\rangle - d|11\rangle$$

Graph States: Construction

vertex $\longleftrightarrow$ qubit in $|+\rangle$ state

edge $\longleftrightarrow$ C operator on ends of the edge vertex $\longleftrightarrow$ qubit in $|+\rangle$ state

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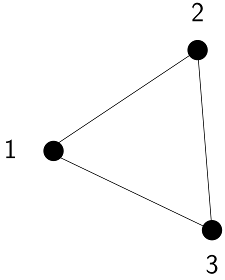
Example:

Graph $G = K_3$



State $|\psi_G\rangle = |K_3\rangle$

$$|000\rangle + |001\rangle + |010\rangle + |100\rangle$$
$$- (|011\rangle + |101\rangle + |110\rangle + |111\rangle)$$

Graph states, cont'd

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- Graph states play a key role in encoding and error correction theory and implementation.

Graph states, cont'd

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Philosophy: local symmetry is important and useful in studying entanglement in general. Among all local operators, the Paulis play a special role for encoding and error correction.

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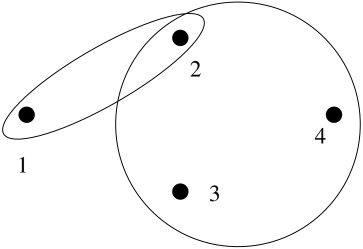
- study natural generalizations of graphs and graph states
- look for algorithm and encoding applications
- study local symmetries, especially Pauli

Hypergraphs

Hypergraph

A hypergraph $G = (V, E)$ is a set V of vertices and a set E of subsets of V. Each $e \in E$ is called a *hyperedge*.



Hypergraph states

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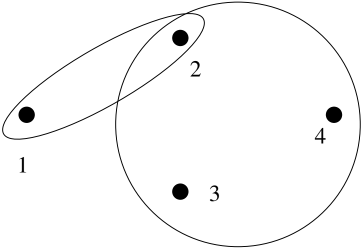

$$
\begin{aligned}
|\psi\rangle &= C_{1,2}C_{2,3,4}\,|+\rangle^{\otimes 4} \\
&= |0000\rangle + |0001\rangle + |0010\rangle + |0011\rangle + |0100\rangle \\
&+ |0101\rangle + |0110\rangle + |1000\rangle + |1001\rangle + |1010\rangle + |1011\rangle + |1111\rangle \\
&- |0111\rangle - |1100\rangle - |1101\rangle - |1110\rangle
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Facts:

- If you have a black box that can decide whether an input graph state is a product state, you can solve 3-SAT

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Question: what hypergraphs G admit local Pauli symmetry? I.e., solve

$$\alpha M_1 \otimes M_2 \otimes \cdots \otimes M_n |\psi_G\rangle = |\psi_G\rangle$$

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Even specialer case: assume G is permutation invariant

Permutation group action on states

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We write $\left|K_n^{m_1,\ldots,m_k}\right\rangle$ to denote the n-qubit hypergraph state that is complete in levels $m_1, \ldots, m_k$ (i.e., has all possible hyperedges of the sizes listed)

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Observation: expansion in standard basis must obey constant coefficients for a given Hamming weight

$$|\psi_G\rangle = \sum_{w=0}^{n} (-1)^{e_w} \sum_{I:\ \mathrm{wt}(I)=w} |I\rangle$$

Permutation invariant hypergraph states, cont'd

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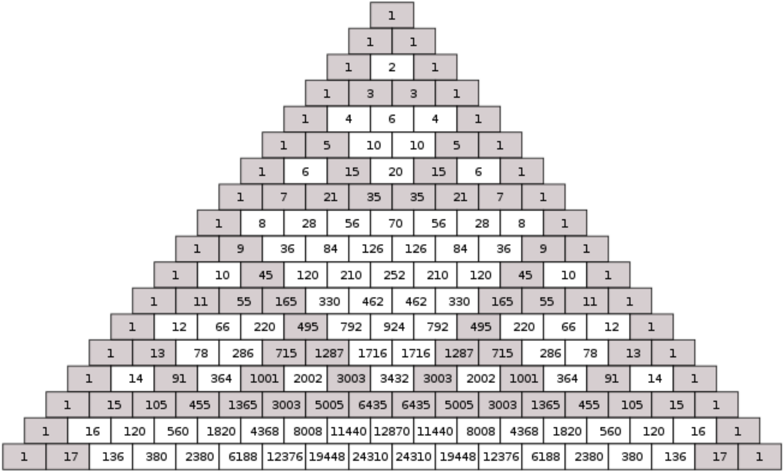
Example(s): read off a list of weight class sign coefficients for one or more $|K_n^m\rangle$ states

Pascal's Triangle mod 2



(image from mathforums.org)

$X^{\otimes n}$ symmetry for perm. inv. hyp. states

Pauli X swaps $0 \leftrightarrow 1$, so $X^{\otimes n}$ take a weight w basis vector to a weight $n-w$ vector with all bits flipped, thus we have $X^{\otimes n}$ symmetry if and only if

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This sends us looking for *palindrome* in $\binom{\cdot}{m}$ stripes (find some examples)

$X^{\otimes n}$ symmetry, cont'd

"Short stripe condition", discovered independently, searching for nonlocality examples

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$-X^{\otimes n}$ symmetry is equivalent to *anti*palindrome condition on long strips, also has short stripe condition, (see examples)

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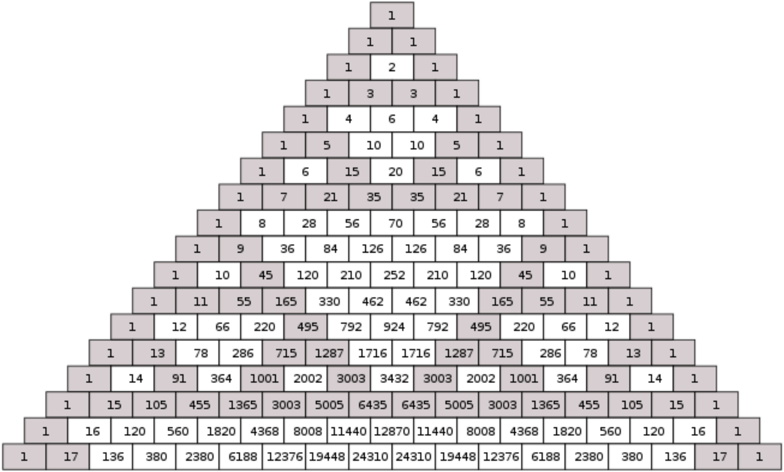

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$X^{\otimes n}$ symmetry, cont'd

For multiple completeness levels $m_1, m_2, \ldots, m_k$, we have

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Summer 2015 work of students, finding (via search) and proving some families

Example: $\left|K_{11+8k}^{2,4}\right\rangle$ has $-X^{\otimes n}$ symmetry for $k \geq 0$

Amusing combinatorics

Two vectors that specify a permutation invariant hypergraph $\left|K_n^{m_1,m_2,\ldots,m_k}\right\rangle$

$$e\colon \;|\psi_G\rangle = \sum_I (-1)^{e_{\mathrm{wt}(I)}} |I\rangle$$

$$g\colon g_w = \begin{cases} 1 & \text{if } m_j = 1 \text{ for some } j \\ 0 & \text{otherwise} \end{cases}$$

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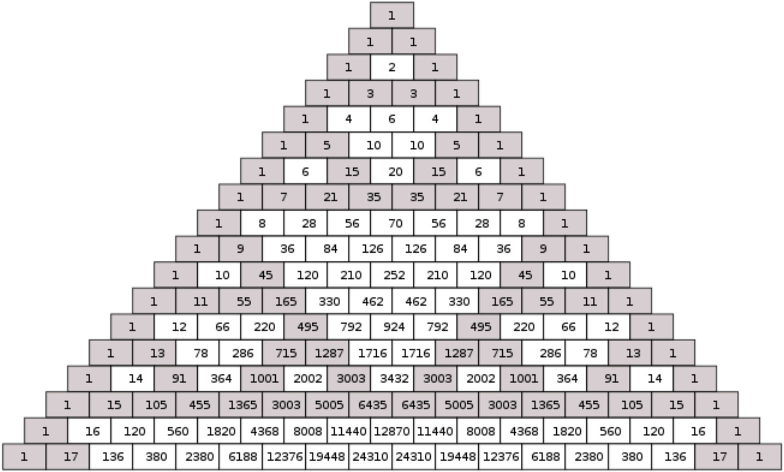
We have $Ae = g$, $Ag = e$. Nice, huh?

Pascal's Triangle mod 2



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Amusing combinatorics, cont'd

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1. Expand n, m in base p, with p^i coefficient digit n_i, m_i, resp.

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1. Expand n, m in base p, with p^i coefficient digit n_i, m_i, resp.
2. $p \mid \binom{n}{m}$ if and only if $\exists i\ m_i > n_i$

Amusing combinatorics, cont'd

Par(i)ty trick: when is $\binom{n}{m}$ even, odd? More generally, when does $p \mid \binom{n}{m}$?

Answer: Kümmer (1852), Lucas (1878).

1. Expand n, m in base p, with p^i coefficient digit n_i, m_i, resp.
2. $p \mid \binom{n}{m}$ if and only if $\exists i \; m_i > n_i$
3. For $p = 2$, $\binom{n}{m}$ is even if and only if there is a position where the base 2 expansion of m has a 1 and the base 2 expansion of n has a 0.

Amusing combinatorics, cont'd

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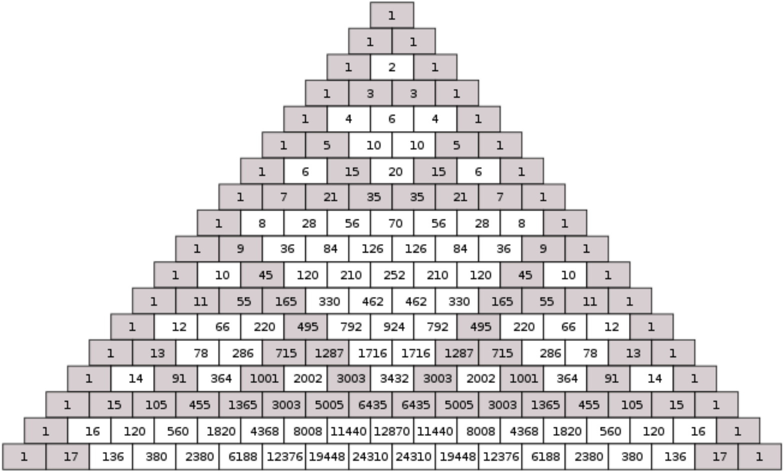
Examples: see Pascal's triangle



(image from mathforums.org)

Amusing combinatorics, cont'd

Question for audience: when is a Pascal row perpendicular to a vector of ± 1 entries? (Besides the one we know, alternating ± 1.) Example in row 14.

Bloch sphere picture

One way to make a permutation invariant state:

1. choose n 1-qubit states $|\psi_1\rangle, \ldots, |\psi_n\rangle$

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Consequence: There is a one-to-one correspondence between n-qubit permutation invariant states and collections of n points on the Bloch sphere.

Bloch sphere picture: it gets better

Local equivalence: Suppose permutation invariant states $|\psi\rangle, |\psi'\rangle$ are locally equivalent. Then there is a 2×2 unitary U such that $|\psi'\rangle = U^{\otimes n} |\psi\rangle$.

Bloch sphere picture: it gets better

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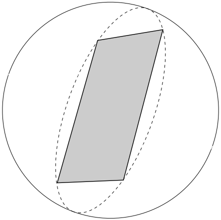
A 2×2 unitary U acts on the Bloch sphere by rotation. So $|\psi\rangle, |\psi'\rangle$ are locally equivalent if and only if their configurations of Bloch points can be rotated one to the other.

Bloch sphere picture, cont'd

Example: $\left|K_4^3\right\rangle$



Bloch configuration are the 4 points at the corners of a rectangle on a great circle, symmetry group is $Z_2 \times Z_2$. Axis of rotations are Y, $\alpha X + \beta Z$, and $-\beta X + \alpha Z$.

Bloch sphere picture, cont'd

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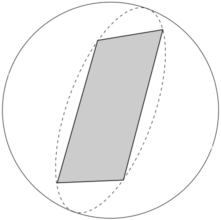

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Conjecture(s)/Question(s): Do all discrete symmetries of permutation invariant hypergraph states have order 2? Are there any axes of symmetry other than X, Y, and these two exotic X, Z-plane axes for $\left|K_4^3\right\rangle$?

Outline

We have found relations between discrete symmetry for perm. inv. hypergraph states with properties of Pascal's triangle mod 2

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We have lots of questions about discrete symmetries

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We would love to develop a killer app for hypergraph states: code(s) with good properties, an algorithm that can be done with hypergraphs but not graphs

Thank you!

Visit us at our website

`http://quantum.lvc.edu/mathphys`