Secondary Characteristic Classes and applications

David L. Johnson

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Abstract

Secondary characteristic classes are geometric invariants, constructed as potentials (antiderivatives) of lifts of primary characteristic forms to an appropriate space, and give invariants either when the primary characteristic form vanishes completely, or as a difference form between two geometric structures. The first appearance of secondary characteristic classes was Chern's proof of the generalized Gauss-Bonnet theorem, but was used by him only in a limited way.

A beautiful paper by Chern and James Simons in 1974 re-constructed Chern's original transgressive forms in a more general setting, and initiated the study of these secondary invariants in their own right. The first non-trivial such invariant was constructed on a compact 3-manifold as a conformal invariant, but has become a fundamental "action integral" in theoretical physics.

I will discuss further applications of these invariants, to:

1. Moduli problems for vector bundles (Jacobians)
2. Boundary terms for characteristic classes
3. Ricci flow

Gauss-Bonnet

In 1830 or so, Gauss (and Bonnet, separately: Bonnet actually published the result in some form; Gauss apparently did not), came up with a local version of a result now named after both of them, which connects the Euler characteristic of a surface to an integral of curvature, relating the geometry of the surface to this basic topological invariant.

Theorem

[Gauss–Bonnet] On a closed, oriented surface Σ,

$$\int_{\Sigma} K dS = 2\pi\chi(\Sigma),$$

where K is the Gaussian (or intrinsic) curvature of the surface and $\chi(\Sigma)$ is the Euler characteristic of the surface, $\chi(\Sigma) = \#faces - \#edges + \#vertices$ for any triangulation of the surface.

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Poincaré-Hopf

if you have a "generic" continuous vector field X tangent to a compact manifold M of arbitrary dimension, then there have to be a finite number of points p_i on the manifold where the vector field is 0, and integers n_i (called the *index* of the vector field at that point), are related to the Euler characteristic, connecting it to differential topology.

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[Poincaré-Hopf] *If X is a continuous tangent vector field on M, with finitely many zeros $\{p_1, \ldots, p_k\}$ of index $\{n_1, \ldots, n_k\}$, then*

$$\sum_i n_i = \chi(M).$$

Near a singular point p_i, divide X by its length, $X/|X|$. For a given distance ϵ from p_i, which is a small $(n-1)$-sphere, you get a map from that S^{n-1}, to the unit tangent vectors, which are also a sphere. Any continuous map from S^{n-1} to itself is determined by its *degree* n_i. That degree is the index of the point.

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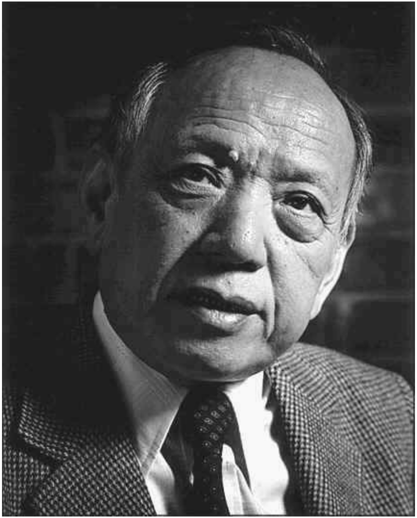

Gauss-Bonnet-Chern

In three remarkable papers from 1944 to 1946, Shiing-Shen Chern set the stage for the modern theory of characteristic classes. The 1945 paper, (*On the curvatura integra in a Riemannian manifold*) was more complex in appearance than the predecessor (*A simple, intrinsic proof of the Gauss-Bonnet theorem for closed Riemannian manifolds*). What each did was show that there was an integrand, made up from the curvature tensor, which when integrated over the manifold gave the Euler characteristic, for any compact, oriented, even-dimensional manifold M^{2n}. The third paper constructed what we now call Chern classes.

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[Allendorfer and Weil, S-S Chern] Let M^{2n} be an even-dimensional, compact, oriented Riemannian manifold. For a specific polynomial $\chi(\Omega)$ of degree n in the terms of the curvature form, the Euler characteristic of M is the integral

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Chern's Proof

Chern constructed an auxiliary form Φ which was of degree $(2n-1)$, but which was not defined on the manifold itself, Φ was defined on the unit tangent bundle to the manifold, $\pi : T_1(M) \to M$. It depended not only on the curvature, but on the connection as well. The form $\chi(M)$ had already been defined by Allendorfer and Weil; it made sense on M, and depended only on curvature (so was tensorial).

The first purpose of that auxiliary form Φ was to show that, on the unit tangent bundle,

$$d\Phi = \pi^*\chi(\Omega),$$

so that it really is a potential form, for a very specific $2n$-form on M. But also, he used this to construct an almost trivial proof that this integrand actually gave the Euler characteristic. Given a generic vector field X on M, except for a finite number of singular points you can make it a unit vector field by $U := X/|X|$, and that $U : M \to T_1(M)$ (except at those points) allows us to pull back this potential equation to M itself. Where you could pull it back, of course $dU^*\Phi = U^*\pi^*\chi(\Omega) = \chi(\Omega)$.

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Explicitly

In this version, ω_{ij} is the connection form for the Levi-Civita connection, with respect to a frame $\{e_1, \ldots, e_n\}$ on $T_1(M)$ so that e_n is the outward-pointing normal at each $\nu \in T_1(M)$. Ω_{ij} is the curvature form. ω_{ij} and Ω_{ij} are skew-symmetric matrices of forms (degree 1 and 2), depending on the frame. Then, (how he saw this, I have no idea), if you set

$$\Phi_k \quad := \quad \sum_{\alpha \in S_{n-1}} (-1)^\alpha \, \Omega_{\alpha_1 \alpha_2} \wedge \cdots \wedge \Omega_{\alpha_{2k-1}\alpha_{2k}} \wedge \omega_{\alpha_{2k+1} n} \wedge \cdots \wedge \omega_{\alpha_{n-1} n},$$

then

$$\Phi = \frac{1}{\pi^p} \sum_{k=0}^{p-1} \frac{1}{1 \cdot 3 \cdots (2p-2k-1) \cdot 2^{p+k} k!} \Phi_k.$$

The form $\chi(\Omega)$ is a little less complicated,

$$\chi(\Omega) = \frac{1}{2^{2p} \pi^p p!} \sum_{\alpha \in S_{2p}} (-1)^\alpha \Omega_{\alpha(1)\alpha(2)} \cdots \Omega_{\alpha(2p-1)\alpha(2p)}.$$

GBC2

Looking carefully at his primitive term Φ, using Stokes' theorem, and cutting small balls $B_{p_i}(\epsilon)$ of radius ϵ away from each of the singular points $\{p_1, \ldots, p_k\}$ of the vector field

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\int_M \chi(\Omega) &= \lim_{\epsilon \downarrow 0} \int_{M - \{B_{p_1}(\epsilon), \ldots, B_{p_k}(\epsilon)\}} \chi(\Omega) \\
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More generally, if M has a boundary, then the boundary has a well-defined unit normal ν, and the theorem becomes (not stated by Chern):

$$\int_M \chi(\Omega) = \chi(M) + \int_{\partial M} \nu^* \Phi.$$

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Classical Invariant Theory

That form $\chi(\Omega)$ is only one of several such invariant tensors defined from the curvature. All of these invariants come from classical linear algebra. Given a $k \times k$ matrix A, an invariant polynomial P is a polynomial expression $P(A)$ in the components of A that is invariant under conjugation of A. Some invariant polynomials are invariants under conjugation by $Gl(k, \mathbb{R})$, like the determinant or the trace, others only are invariant under subgroups and defined for related matrices.

Groups and invariants

$U(k)$ $A \in u(k)$, complex $k \times k$ matrices so that $\overline{A}^t = -A$. Any coefficient of the complex characteristic polynomial $det_{\mathbb{C}}(A - \lambda I)$ is an invariant polynomial in A. Up to a constant, the coefficient of λ^{k-j} is the j^{th} Chern polynomial $c_j(A)$.

$O(k)$ $A \in o(k)$, skew-symmetric $n \times n$ matrices. Characteristic polynomial $\det_{\mathbb{R}}(A - \lambda) = \prod(r_i^2 + \lambda^2)$, even degrees only; generate Pontryagin polynomials $p_j(A)$ (degree $2j$).

$SO(2k)$ $A \in o(2k)$, but orientation taken into account. Get one extra polynomial $e(A) = \sqrt{\det(A)} = \mathit{Pfaff}(A)$.

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Geometries

A vector bundle $E \to M$ (such as the tangent bundle) has various structural groups associated with it. Geometric structures correspond to certain structural groups: if E is a complex vector bundle, rank k, with a compatible metric has $U(k)$ as structural group. A real vector bundle has $O(k)$, or, if oriented, $SO(k)$.

Corresponding to each geometric structure, you have connection and curvature forms with values in the Lie algebras of those groups, $u(k)$ (skew-Hermitian), $o(k)$, and $so(k)$. So, applying the invariant polynomials to the curvature, you get forms:

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Euler, Chern and Pontryagin

Chern classes: $c_j(\Omega)$, of degree $2j$ for complex bundles of rank k. These classes are a nearly complete topological characterization of a complex vector bundle, giving in particular primary obstructions to the existence of $(k - j + 1)$ frames over $2j$-dimensional submanifolds.

Pontryagin classes, $p_j(\Omega)$, degree $4j$, for real bundles (here the obstruction theory is more complicated), and

Euler class, $e(\Omega)$, of degree $k/2$, for even-rank orientable real bundles. For the tangent bundle, this is $\chi(\Omega)$. This class provides a primary obstruction to the existence of a section of the bundle over $2k$-dimensional submanifolds.

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Chern-Simons

In 1974, Chern and Jim Simons published a paper which connected all of these characteristic classes back to Chern's construction, defining a general formula for the transgression (potential) form and then examining the geometric properties of the potential. To each invariant polynomial P, they constructed a form $TP(\omega)$, which is only defined on the principal bundle of bases of $E \to M$. Explicitly,

$$TP(\omega) \;=\; j \int_0^1 P\left(\omega, \phi_t^{j-1}\right) dt,$$

where ω is the connection form, $\phi_t := t\Omega + (t^2 - t)\,[\omega, \omega]$ is a mysterious 2-form, and the polynomial $P(A)$ is turned into a multilinear symmetric function by polarization.

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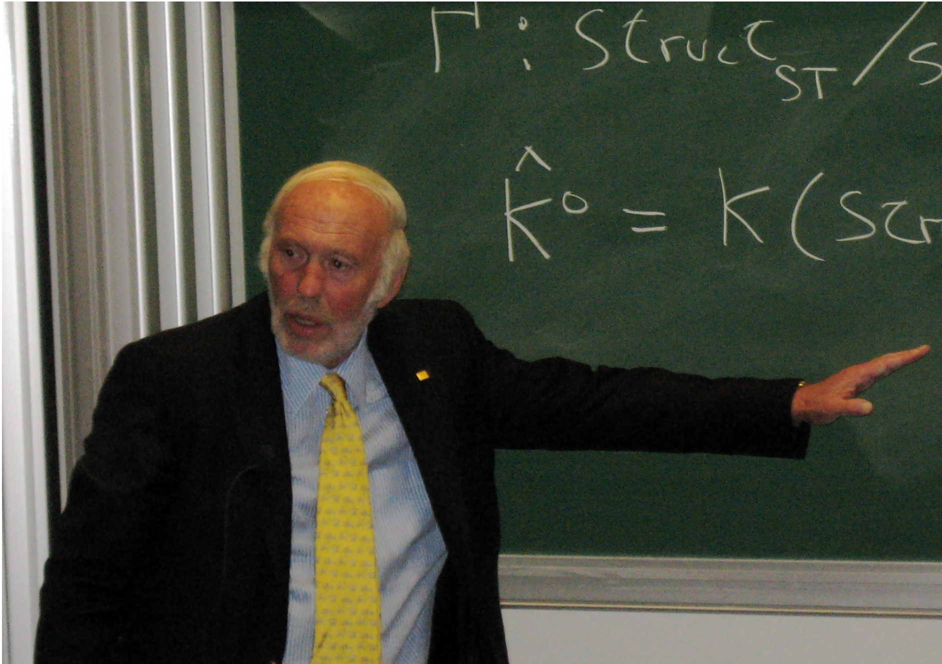
Theorem

[Chern and Simons] *For any degree-j invariant polynomial P of matrices $A \in \mathfrak{g}$, $\mathfrak{g}$ a Lie algebra, and any principal G-bundle $\pi : B \to M$, with connection ω and curvature Ω, the form $TP(\omega)$ satisfies*

$$dTP(\omega) \quad = \quad \pi^* \left(P(\Omega)\right).$$

Simons



The point of their construction is in the applications. If, for example, for some connection ω and curvature Ω the primary class $P(\Omega) = 0$ (as a form, not just in cohomology), then $dTP(\omega) = 0$, so we get a cohomology class, a secondary characteristic class. This is still on the principal bundle B, but modulo integer cohomology (forms of integral periods), it is well-defined as a cohomology class in $H^{2j-1}(M, \mathbb{R}/\mathbb{Z})$. That class does depend on the connection, so gives geometric, not topological, information.

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3-manifolds

Let M^3 be a compact, oriented Riemannian 3-manifold. The bundle we consider is the tangent bundle, with the connection being the Levi-Civita connection of a given metric on the manifold. Then, since $p_1(\Omega)$, the form representing the first Pontryagin class, is a 4-form on a 3-manifold, it is 0. Even when lifted up to the bundle of frames of the tangent bundle, still 0. So, $Tp_1(\omega)$ is a well-defined element of $H^3(M, \mathbb{R}/\mathbb{Z}) = \mathbb{R}/\mathbb{Z}$. But, since $T_*(M)$ is trivial, for a given trivialization $Tp_1(\omega)$ can be defined as an element of $H^3(M, \mathbb{R}) = \mathbb{R}$. The integral $\int_M Tp_1(\omega)$ is called the Chern-Simons invariant (or action, or energy) in the theoretical physics literature, and is interpreted as having physical significance.

Chern and Simons interpret the invariant slightly differently, as a conformal invariant:

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Conformal Metrics

Theorem

[Chern-Simons] *If M_t is a family of 3-manifolds that are conformally equivalent, then*

$$\frac{d}{dt} Tp_1(M_t) = 0,$$

so the Chern-Simons class is a conformal invariant.

They also have a corollary which would seem to be somewhat more intrinsically interesting:

Theorem

If M^3 is simply-connected, then for all Riemannian metrics M_g, define $\Phi(g) := \int_M Tp_1(M_g)$. Either Φ has exactly one critical point and $M \cong S^3$, or Φ has no critical point and $M \not\cong S^3$.

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Ricci Flow

Ricci flow is a particular evolution of a Riemannian metric, given as a solution to the differential equation

$$\frac{\partial g_{ij}}{\partial t} = -2Ric_{ij},$$

so the change of the metric with respect to time is in the direction of the Ricci tensor. I wondered whether there was any relationship between this flow and the Chern-Simons class above, and had my student, Chris Godbout, work on the problem of whether $Tp_1(\omega)$ was invariant along a Ricci flow, or perhaps varied with the Ricci flow.

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Complex vector bundles

If M is a Kähler manifold, and $E \to M$ is a complex vector bundle, then automatically E is an almost-complex manifold. It may be that there is an integrable complex structure (so that $\pi : E \to M$ is holomorphic), or maybe not, but if there are there may be many inequivalent such structures, different holomorphic bundles with the same underlying topological structure. Secondary characteristic classes play a role in understanding these moduli problems.

Let $\mathcal{M}(E_0) = \{(E_0, J)/\sim\}$ be the set of holomorphic bundles realizable on a given bundle E_0 (with complex structure J_0), modulo holomorphic equivalence. Fixing a Hermitian inner product on E_0 (it will be Hermitian on all (E_0, J)) determines a connection ω on the bundle of bases for each (E_0, J), along with a curvature form Ω that is a matrix of forms of bidegree $(1,1)$ $[dz_j \wedge d\bar{z}_j]$. Then, for any Chern polynomial c_j, the difference forms

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are well-defined on the bundle of bases, and, projected to the forms of bidegrees (p, q) with $p < q$ become closed in that complex of forms, and are well-defined modulo the images of integer-valued forms.

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Intermediate Jacobians

Theorem

The mapping

$$Tc_j : \mathcal{M}(E_0) \quad \to \quad H^{0,2p-1}(M) \oplus \cdots \oplus H^{p-1,p}(M)/H^{2p-1}(M,\mathbb{Z})$$

defined by $(E, J) \mapsto [Tc_j(\omega) - Tc_j(\omega_0)]$ *gives a map from* $\mathcal{M}(E_0)$ *to the (Griffiths) intermediate Jacobian.*

Theorem

If E_0 *has rank 1, the mapping*

$$Tc_1 : \mathcal{M}(E_0) \quad \to \quad H^{0,1}(M)/H^1(M,\mathbb{Z}) \cong Pic_0(M)$$

defined by $(E, J) \mapsto [Tc_1(\omega) - Tc_1(\omega_0)]$ *is an isomorphism.*

Intermediate Jacobians

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Associated bundles

You may have noticed a significant difference between this new theory and Chern's original results, which after all started all this. The forms $TP(\omega)$ are always defined on the principal bundle of bases of the vector bundle, not on the vector bundle itself, or on its unit sphere bundle, while Chern's form Φ was defined on the unit tangent bundle. In general if $B \to M$ is a fiber bundle with fibers (say) a homogeneous space $F = G/H$, then there is associated to that bundle a principal bundle $P \to M$. $P \to B$ is also a principal bundle, with fiber H. Any connection ω on P can be split up to those parts which are vertical (tangent to the fibers H of $P \to B$ and those which are horizontal, $\omega = \phi + \psi$. Similarly, for any polynomial Q, there is a decomposition of the potential form $TQ(\omega)$ into parts that are well-defined on $B \to M$ and those which are only defined on $P \to B \to M$.

The part which is defined on $B \to M$, denoted $\Phi Q(\omega)$, satisfies a more complicated equation than would be suggested by Chern's original result, even though that is a special case of this theorem. Another special case of this was called a **heterotic** formula by Vafa-Whitten (also called an anomaly equation):

Theorem

[-, Nie] *If $\pi : B \to M$ is an associated bundle with fiber $F = G/H$, and if $p : P \to M$ is the associated principal bundle, with $\pi_1 : P \to B$ the principal H-bundle connecting them, then for any invariant polynomial Q of G the form $\Phi Q(\omega)$ satisfies*

$$d\Phi Q(\omega) = Q(\Omega) - Q(\Psi),$$

where Ψ is the curvature of $P \to B$.

In some cases, such as if Q is the Euler polynomial and the associated bundle is the unit tangent bundle, the form $Q(\Psi)$ will be identically 0, recovering Chern's result. This also immediately gives boundary terms for not only the Euler class, but for the Chern classes, providing a direct, rather than "universal" proof of the basic obstruction properties.

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More secondary classes

In any of the above situations, if $Q(\Omega) = 0$, such as if the dimension of M is $2deg(Q) - 1$, there will be corresponding secondary invariants in the $\mathbb{R}/\mathbb{Z}$ cohomolgy of the base. The Pontryagin classes are more complicated, in general, than the others, but all are well-defined.

Theorem

If M^{4k-1} is a compact, simply-connected manifold, with $B \to M$ the unit tangent bundle of M, and if ω is the Riemannian connection of a given Riemannian metric on M, then the forms $\sigma^*(\Phi p_k(\omega))$ are well-defined as secondary characteristic classes in $H^{4k-1}(M, \mathbb{R})$, depending only upon the metric, and are conformally invariant.

Fact

These forms are not always the same as the Chern-Simons forms.

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