Linking integrals in three-dimensional geometries

D. DeTurck

University of Pennsylvania

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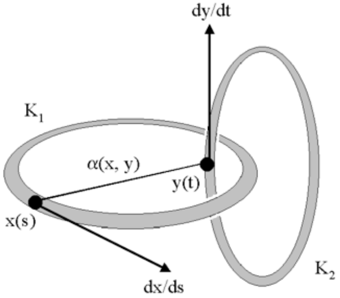
Gauss linking integral

Carl Friedrich Gauss, in a half-page paper dated January 22, 1833, gave an integral formula for the linking number in Euclidean 3-space,

$$\mathrm{Link}(K_1, K_2) = \int_{K_1 \times K_2} \frac{d\mathbf{x}}{ds} \times \frac{d\mathbf{y}}{dt} \cdot \frac{\mathbf{x} - \mathbf{y}}{4\pi|\mathbf{x} - \mathbf{y}|^3}\, ds\, dt.$$



General goal

Our goal is to define geometrically natural linking integrals for each of the eight homogeneous three-dimensional geometries

$$\mathbb{R}^3, \quad S^3, \quad H^3, \quad S^2 \times \mathbb{R}, \quad H^2 \times \mathbb{R},$$

$$\text{Nil}, \quad \text{Sol}, \quad SL(2, \mathbb{R})$$

and at least some of their higher-dimensional generalizations. "Geometrically natural" in this context means that the integrands should be invariant under orientation-preserving isometries of the ambient spaces.

Another expression for the Gauss integral

Another way to write Gauss's formula is to define the double-form

$$\Phi_{1,1}(\mathbf{x},\mathbf{y}) = \frac{\omega_{1,1}}{4\pi|\mathbf{x}-\mathbf{y}|^3}$$

on $\mathbb{R}^3 \times \mathbb{R}^3$, where

$$\begin{aligned}\omega_{1,1} = (y_1 - x_1)(dx_2 \otimes dy_3 - dx_3 \otimes dy_2) + (y_2 - x_2)(dx_3 \otimes dy_1 - dx_1 \otimes dy_3) \\ + (y_3 - x_3)(dx_1 \otimes dy_2 - dx_2 \otimes dy_1).\end{aligned}$$

Thinking of the curves K_1 and K_2 as maps from S^1 into $\mathbb{R}^3$, we define

$$\mathrm{Link}(K_1, K_2) = \iint_{S^1 \times S^1} \mathbf{X}^* \Phi_{1,1}$$

where $\mathbf{X} = (\mathbf{x}, \mathbf{y})$ is the product mapping.

Double forms

- A *double-form* is a differential form on $M \times M$ which can be viewed either as a differential form on the first factor with coefficients being differential forms on the second factor, or vice versa.
- A (p, q)-form has of degree p over the first M factor and of degree q over the second. For example, a $(2, 1)$-form on $\mathbb{R}^3 \times \mathbb{R}^3$ can be expressed as:

$$f(\mathbf{x}, \mathbf{y}) dx_2 \wedge dx_3 \otimes dy_1 + \cdots$$

  and so on for nine terms.
- For such forms, we have exterior derivatives $d_{\mathbf{x}}$ and $d_{\mathbf{y}}$ which commute with each other and have other standard properties such as $d_{\mathbf{x}}^2 = d_{\mathbf{y}}^2 = 0$, etc.

Linking integrals

We wish to calculate the linking number of K and L via an integral of the form

$$\mathrm{Link}(K, L) = \int_{K \times L} \Phi_{1,1}(\mathbf{x}, \mathbf{y}),$$

where $\mathbf{x} \in K$ and $\mathbf{y} \in L$ and $\Phi_{1,1}$ is an appropriately-chosen isometry-invariant (1,1)-form on $(M) \times (M)$. The form $\Phi_{1,1}$ will be singular along the diagonal Δ of $(M) \times (M)$, but will be smooth otherwise.

Link-homotopy invariance

If there is a (2,0)-form $\Psi_{2,0}(\mathbf{x}, \mathbf{y})$ having the property that $d_{\mathbf{y}}\Psi_{2,0} = d_{\mathbf{x}}\Phi_{1,1}$ as (2,1)-forms on $M \times M) \setminus \Delta$, define the ordinary 1-form

$$\Omega_1(\mathbf{x}) = \int_L \Phi_{1,1}(\mathbf{x}, \mathbf{y})$$

on $M \setminus L$ by integrating $\Phi_{1,1}$ over the curve L for each $\mathbf{x} \notin L$. Then we will have

$$d_{\mathbf{x}}\Omega_1 = d_{\mathbf{x}} \int_L \Phi_{1,1}(\mathbf{x}, \mathbf{y}) = \int_L d_{\mathbf{x}}\Phi_{1,1}(\mathbf{x}, \mathbf{y}) = \int_L d_{\mathbf{y}}\Psi_{2,0}(\mathbf{x}, \mathbf{y}) = 0,$$

by Stokes's Theorem. So Ω_1 is a closed 1-form (in $\mathbf{x}$) on $M \setminus L$, and the value of

$$\int_K \Omega_1(\mathbf{x}) = \int_{K \times L} \Phi_{1,1}(\mathbf{x}, \mathbf{y}) = \mathrm{Link}(K, L)$$

depends only on the homology class of K within $M \setminus L$, so K can be deformed without affecting the value of the linking integral as long as K never meets L.

Link-homotopy invariance

- Likewise, if there is a (0,2)-form $\Psi_{0,2}(\mathbf{x}, \mathbf{y})$ having the property that $d_{\mathbf{x}}\Psi_{0,2} = d_{\mathbf{y}}\Phi_{1,1}$ as (1,2)-forms on $(M \times M) \setminus \Delta$ we can deform L as long as it never meets K and not affect the value of the linking integral.

Link-homotopy invariance

- Likewise, if there is a (0,2)-form $\Psi_{0,2}(\mathbf{x}, \mathbf{y})$ having the property that $d_{\mathbf{x}}\Psi_{0,2} = d_{\mathbf{y}}\Phi_{1,1}$ as (1,2)-forms on $(M \times M) \setminus \Delta$ we can deform L as long as it never meets K and not affect the value of the linking integral.
- Our goal will be to produce isometry-invariant forms $\Phi_{1,1}$, $\Psi_{2,0}$ and $\Psi_{0,2}$ and to show that they satisfy the differential equations $d_{\mathbf{y}}\Psi_{2,0} = d_{\mathbf{x}}\Phi_{1,1}$ and $d_{\mathbf{x}}\Psi_{0,2} = d_{\mathbf{y}}\Phi_{1,1}$.

$\mathbb{R}^n$, S^n, H^n

Write $\mathbf{x} = (x_0, x_1, \ldots, x_n)$ and $\mathbf{y} = (y_0, y_1, \ldots, y_n)$ for points in $\mathbb{R}^{n+1}$, and $\langle \mathbf{x}\,,\,\mathbf{y}\rangle_\pm$ for the inner product on $\mathbb{R}^{n+1}$ given by

$$\langle \mathbf{x}\,,\,\mathbf{y}\rangle_\pm = x_0 y_0 \pm (x_1 y_1 + x_2 y_2 + \cdots + x_n y_n).$$

We will then view $S^n \subset \mathbb{R}^{n+1}$ as

$$S^n = \{\mathbf{x} \in \mathbb{R}^{n+1} \mid \langle \mathbf{x}\,,\,\mathbf{x}\rangle_+ = 1\},$$

$H^n \subset \mathbb{R}^{n+1}$ as

$$H^n = \{\mathbf{x} \in \mathbb{R}^{n+1} \mid \langle \mathbf{x}\,,\,\mathbf{x}\rangle_- = 1,\ x_0 > 0\},$$

and $\mathbb{R}^n \subset \mathbb{R}^{n+1}$ as

$$\mathbb{R}^n = \{\mathbf{x} \in \mathbb{R}^{n+1} \mid x_0 = 1\}.$$

Group actions

There are natural group actions on $\mathbb{R}^{n+1}$ that restrict to groups of isometries on S^n, H^n and $\mathbb{R}^n$, namely the standard actions of $SO(n+1)$, $SO_+(1,n)$ and $E(n)$ respectively, where $E(n)$ is the group of Euclidean motions of $\mathbb{R}^n$. An element of $E(n)$ is given by the matrix

$$\left[\begin{array}{c|c} 1 & \mathbf{0} \\ \hline \mathbf{v} & R \end{array}\right],$$

where $R \in SO(n)$ and $\mathbf{v} \in \mathbb{R}^n$, so this matrix rotates $\mathbb{R}^n$ according to the matrix R and then translates by $\mathbf{v}$.

Basic invariant forms

Let $\mathbf{x}$ and $\mathbf{y}$ be two points in $\mathbb{R}^{n+1}$. For each k and ℓ satisfying $k + \ell = n - 1$, define a differential form $\omega_{k,\ell}$ as follows:

For $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \in T_{\mathbf{x}}\mathbb{R}^{n+1}$ and $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_\ell \in T_{\mathbf{y}}\mathbb{R}^{n+1}$,

$$\omega_{k,\ell}(\mathbf{x}, \mathbf{y}; \mathbf{v}_1, \ldots, \mathbf{v}_k; \mathbf{w}_1, \ldots, \mathbf{w}_\ell) = \|\mathbf{x}, \mathbf{y}, \mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{w}_1, \ldots, \mathbf{w}_\ell\| .$$

Formally think of $\omega_{k,\ell}$ as the determinant

$$\frac{1}{k!\ell!} \|\mathbf{x}, \mathbf{y}, d\mathbf{x}, \ldots, d\mathbf{x}, d\mathbf{y}, \ldots, d\mathbf{y}\|$$

More forms

Two other families of forms for $k + \ell = n$:

$$\alpha_{k,\ell}(\mathbf{x}, \mathbf{y}; \mathbf{v}_1, \ldots, \mathbf{v}_k; \mathbf{w}_1, \ldots, \mathbf{w}_\ell) = \|\mathbf{x}, \mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{w}_1, \ldots, \mathbf{w}_\ell\|,$$

$$\beta_{k,\ell}(\mathbf{x}, \mathbf{y}; \mathbf{v}_1, \ldots, \mathbf{v}_k; \mathbf{w}_1, \ldots, \mathbf{w}_\ell) = \|\mathbf{y}, \mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{w}_1, \ldots, \mathbf{w}_\ell\|.$$

Forms restrict in a natural way to M, are invariant under the action of G. Also write:

$$\alpha_{k,\ell} = \frac{1}{k!\ell!} \|\mathbf{x}, d\mathbf{x}, \ldots, d\mathbf{x}, d\mathbf{y}, \ldots, d\mathbf{y}\|$$

$$\beta_{k,\ell} = \frac{1}{k!\ell!} \|\mathbf{y}, d\mathbf{x}, \ldots, d\mathbf{x}, d\mathbf{y}, \ldots, d\mathbf{y}\|.$$

Define exterior differential operators, $d_\mathbf{x}$ and $d_\mathbf{y}$. Check that

$$d_\mathbf{x}\omega_{k,\ell} = \frac{1}{k!\ell!} \|d\mathbf{x}, \mathbf{y}, d\mathbf{x}, \ldots, d\mathbf{x}, d\mathbf{y}, \ldots, d\mathbf{y}\| = -(k+1)\beta_{k+1,\ell}$$

and

$$d_\mathbf{y}\omega_{k,\ell} = \frac{1}{k!\ell!} \|\mathbf{x}, d\mathbf{y}, d\mathbf{x}, \ldots, d\mathbf{x}, d\mathbf{y}, \ldots, d\mathbf{y}\| = (-1)^k(\ell+1)\alpha_{k,\ell+1}.$$

Distance functions

- Let $\sigma = \sigma(\mathbf{x}, \mathbf{y}) = \langle \mathbf{x}\,,\, \mathbf{y} \rangle_+$. The geodesic distance α between two points $\mathbf{x}$ and $\mathbf{y}$ on S^n is $\alpha = \arccos \sigma$.
- Likewise, if $\sigma = \langle \mathbf{x}\,,\, \mathbf{y} \rangle_-$, the geodesic distance α between two points $\mathbf{x}$ and $\mathbf{y}$ on H^n is $\alpha = \operatorname{arccosh} \sigma$.
- And of course the distance between two points $\mathbf{x}$ and $\mathbf{y}$ on $\mathbb{R}^n$ is $\alpha = ((\mathbf{y} - \mathbf{x}) \cdot (\mathbf{y} - \mathbf{x}))^{1/2}$ and we let $\sigma = \frac{1}{2}(\mathbf{y} - \mathbf{x}) \cdot (\mathbf{y} - \mathbf{x})$ in this case.

Linking integrands

To construct linking integrands, need an identity of the form:

$$d_{\mathbf{x}}[\varphi(\sigma)\omega_{1,1}] = d_{\mathbf{y}}[\psi(\sigma)\omega_{2,0}],$$

where σ is a function of the distance $\alpha(\mathbf{x}, \mathbf{y})$ between $\mathbf{x}$ and $\mathbf{y}$. On S^3 and H^3, use $\sigma = \langle \mathbf{x}\,,\, \mathbf{y} \rangle_{\pm}$, so that $\sigma = \cos\alpha$ and $\sigma = \cosh\alpha$ respectively.
On S^3 (and H^3), we have

$$d_{\mathbf{x}}[\varphi(\sigma)\omega_{1,1}] = \varphi'(\sigma)\,[\alpha_{2,1} - \sigma\beta_{2,1}] - 2\varphi(\sigma)\beta_{2,1}$$

and

$$d_{\mathbf{y}}[\psi(\sigma)\omega_{2,0}] = -\psi'(\sigma)\,[\beta_{2,1} - \sigma\alpha_{2,1}] + \psi(\sigma)\alpha_{2,1}.$$

Differential equations for φ and ψ

Because $\alpha_{2,1}$ and $\beta_{2,1}$ are independent we need the coefficients of both to be equal for these quantities to be equal. So φ and ψ must satisfy:

$$\varphi'(\sigma) - \sigma\psi'(\sigma) - \psi(\sigma) = 0$$
$$\sigma\varphi'(\sigma) + 2\varphi(\sigma) - \psi'(\sigma) = 0.$$

Derive that φ satisfies the second-order equation

$$(1-\sigma^2)\varphi'' - 5\sigma\varphi' - 4\varphi = 0. \qquad (*)$$

Likewise

$$(1-\sigma^2)\psi'' - 5\sigma\psi' - 3\psi = 0.$$

Thus we can find functions ψ and χ so that

$$d_{\mathbf{x}}[\varphi\omega_{1,1}] = d_{\mathbf{y}}[\psi\omega_{2,0}] \qquad \text{and} \qquad d_{\mathbf{y}}[\varphi\omega_{1,1}] = d_{\mathbf{x}}[\chi\omega_{0,2}].$$

We refer to this fact as the *Key Lemma*.

Solution on $\mathbb{R}^3$

On $\mathbb{R}^3$ the differential equation for $\varphi(\sigma)$ analogous to $(*)$ is:

$2\sigma\varphi'' + 5\varphi' = 0$ which has general solution $\varphi(\sigma) = \dfrac{C_1}{\sigma^{3/2}} + C_2$.

We want φ to decay to zero when $\sigma \to \infty$, so $C_2 = 0$.

By considering a single nontrivial example, we get that $C_1 = 1/\operatorname{vol}(S^2)$ so

$$\operatorname{Link}(K, L) = \frac{1}{4\pi} \iint_{K \times L} \frac{\omega_{1,1}}{\sigma^{3/2}}$$

just as Gauss did.

Solution on S^3

On S^3,

$$\varphi = c_1 \frac{\sigma}{(1-\sigma^2)^{3/2}} + c_2 \frac{\sqrt{1-\sigma^2} - \sigma \arccos \sigma}{(1-\sigma^2)^{3/2}}.$$

We need

$$\lim_{\sigma \downarrow -1} \varphi(\sigma)$$

to be finite, and considering a single non-trivial example (two orthogonal linked great circles), we conclude that

$$c_2 = \varphi(0) = \frac{1}{4\pi^2} \quad \text{and} \quad c_1 = \frac{1}{4\pi}.$$

This determines the linking integral on S^3.

$H^2 \times \mathbb{R}$

Now consider $M = H^2 \times \mathbb{R}$, and write $\langle \mathbf{v}\,,\, \mathbf{w} \rangle$ rather than $\langle \mathbf{v}\,,\, \mathbf{w} \rangle_- = v_0 w_0 - v_1 w_1 - v_2 w_2$ for the Minkowski inner product on $\mathbb{R}^3$.

View $H^2 \subset \mathbb{R}^3$ as the set $\{\bar{\mathbf{x}}(x_0, x_1, x_2) \,|\, \langle \bar{\mathbf{x}}\,,\, \bar{\mathbf{x}} \rangle = 1\}$. Write the induced inner product on H^2 as $\mathbf{v} \cdot \mathbf{w} = -\langle \bar{\mathbf{v}}\,,\, \bar{\mathbf{w}} \rangle$.

$H^2 \times \mathbb{R}$ is diffeomorphic to $\mathbb{R}^3$.

Isometry group

The connected component of the isometry group of $H^2 \times \mathbb{R}$ is the product of those of H^2 and $\mathbb{R}$, namely $G = SO_+(1,2) \times \mathbb{R}$.

Recall that $SO_+(1,2)$ is the three-dimensional *restricted Lorentz group*, i.e., the Lie group of transformations of $\mathbb{R}^3$ which preserve the indefinite inner product we are using (hence the 'O' and the '1,2'), and which both preserve the orientation of $\mathbb{R}^3$ (hence the 'S') and which independently preserve the orientation of space and the direction of "time" (i.e., of x_0, hence the '$+$'). The $\mathbb{R}$ factor of G acts by translation on the $\mathbb{R}$ factor of $H^2 \times \mathbb{R}$.

The the isometry group acts transitively on $H^2 \times \mathbb{R}$, however, since the group is only 4-dimensional, it cannot act transitively on the unit tangent bundle of $H^2 \times \mathbb{R}$, which is 5-dimensional.

Invariant functions

The distance $D(\mathbf{x}, \mathbf{y})$ between two points $\mathbf{x}$ and $\mathbf{y}$ in $H^2 \times \mathbb{R}$ satisfies

$$D^2 = (\operatorname{arccosh} \langle \bar{\mathbf{x}} \,,\, \bar{\mathbf{y}} \rangle)^2 + (x_3 - y_3)^2.$$

On $H^2 \times \mathbb{R}$, two independent isometry-invariant two-point functions are

$$\sigma(\mathbf{x}, \mathbf{y}) = \langle \bar{\mathbf{x}} \,,\, \bar{\mathbf{y}} \rangle = x_0 y_0 - x_1 y_1 - x_2 y_2 \quad \text{and} \quad \tau(\mathbf{x}, \mathbf{y}) = x_3 - y_3.$$

Any isometry-invariant two-point function can be expressed as $F(\sigma(\mathbf{x}, \mathbf{y}), \tau(\mathbf{x}, \mathbf{y}))$. For instance, the distance $D(\mathbf{x}, \mathbf{y})$ between the two points $\mathbf{x}$ and $\mathbf{y}$ on M satisfies $D^2 = (\operatorname{arccosh} \sigma)^2 + \tau^2$, as noted above.

Key Lemma

Since $((H^2 \times \mathbb{R}) \times (H^2 \times \mathbb{R})) \setminus \Delta$ is simply connected, we have

A necessary and sufficient condition for the integral

$$\int_{K \times L} \Phi_{1,1}$$

to be link-homotopy invariant (i.e., to remain unchanged under deformations of the simple closed curves K and L which keep K and L disjoint) is that $\Phi_{1,1}$ be smooth and

$$d_{\mathbf{y}} d_{\mathbf{x}} \Phi_{1,1} = 0$$

on $((H^2 \times \mathbb{R}) \times (H^2 \times \mathbb{R})) \setminus \Delta$.

Invariant forms

We assume that

$$\Phi_{1,1} = f(\sigma,\tau)\,\omega_{1,0}\otimes dy_3 + g(\sigma,\tau)\,dx_3\otimes\omega_{0,1} + p(\sigma,\tau)\,\alpha_{1,1} + q(\sigma,\tau)\,\beta_{1,1},$$

where $\sigma = \langle \bar{\mathbf{x}}\,,\,\bar{\mathbf{y}}\rangle$ and $\tau = x_3 - y_3$ are the invariant functions on $(H^2\times\mathbb{R})\times(H^2\times\mathbb{R})$. We calculate:

$$\begin{aligned}
d_{\mathbf{y}} d_{\mathbf{x}} \Phi_{1,1} &= d_{\mathbf{y}} d_{\mathbf{x}} (f\,\omega_{1,0}\otimes dy_3 + g\,dx_3\otimes\omega_{0,1} + p\,\alpha_{1,1} + q\,\beta_{1,1}) \\
&= \big((1-\sigma^2) f_{\sigma\sigma} - 4\sigma f_\sigma - 2f + \sigma\, p_{\sigma\tau} + 2p_\tau + q_{\sigma\tau}\big)\,\gamma_{2,1}\wedge dy_3 \\
&\quad + \big((\sigma^2-1) g_{\sigma\sigma} + 4\sigma g_\sigma + 2g + p_{\sigma\tau} + \sigma\, q_{\sigma\tau} + 2q_\tau\big)\,\gamma_{1,2}\wedge dx_3 \\
&\quad + (\sigma f_{\sigma\tau} + f_\tau + g_{\sigma\tau} - p_{\tau\tau})\alpha_{1,1}\wedge dx_3\wedge dy_3 \\
&\quad - (f_{\sigma\tau} + \sigma g_{\sigma\tau} + g_\tau + q_{\tau\tau})\beta_{1,1}\wedge dx_3\wedge dy_3.
\end{aligned}$$

Differential equations

From this we have

Let

$$\Phi_{1,1} = f\,\omega_{1,0} \otimes dy_3 + g\,dx_3 \otimes \omega_{0,1} + p\,\alpha_{1,1} + q\,\beta_{1,1}$$

In order for the linking integral to be link-homotopy invariant we need

$$\begin{aligned}
0 &= (1-\sigma^2)f_{\sigma\sigma} - 4\sigma f_\sigma - 2f + \sigma\,p_{\sigma\tau} + 2p_\tau + q_{\sigma\tau} \\
0 &= (1-\sigma^2)g_{\sigma\sigma} - 4\sigma g_\sigma - 2g - p_{\sigma\tau} - \sigma\,q_{\sigma\tau} - 2q_\tau \\
0 &= \sigma f_{\sigma\tau} + f_\tau + g_{\sigma\tau} - p_{\tau\tau} \\
0 &= f_{\sigma\tau} + \sigma g_{\sigma\tau} + g_\tau + q_{\tau\tau}
\end{aligned}$$

on $((H^2 \times \mathbb{R}) \times (H^2 \times \mathbb{R})) \setminus \Delta$.

Geometric strategy for solving the system

If K is given by $\mathbf{x}(s)$ and L by $\mathbf{y}(t)$ then for each pair of points $\mathbf{x}(s)$,$\mathbf{y}(t)$ we construct the unit "pointing" vectors $\mathbf{p_{yx}}(s,t) \in T_\mathbf{x}(H^2 \times \mathbb{R})$ and $\mathbf{p_{xy}}(s,t) \in T_\mathbf{y}(H^2 \times \mathbb{R})$ that point along the geodesic from $\mathbf{x}(s)$ to $\mathbf{y}(t)$ and from $\mathbf{y}(t)$ to $\mathbf{x}(s)$ respectively.

Calculate the projection of the area spanned by $\partial\mathbf{p_{yx}}/\partial s$ and $\partial\mathbf{p_{yx}}/\partial t$ onto the tangent plane at $\mathbf{p_{yx}}$ to the unit sphere in $T_\mathbf{x}(H^2 \times \mathbb{R})$, and the projection of the area spanned by $\partial\mathbf{p_{xy}}/\partial t$ and $\partial\mathbf{p_{xy}}/\partial s$ onto the tangent plane at $\mathbf{p_{xy}}$ to the unit sphere in $T_\mathbf{y}(H^2 \times \mathbb{R})$.

The average of these gives the value of our linking integral candidate evaluated at $(\mathbf{x},\mathbf{y}) \in (H^2 \times \mathbb{R}) \times (H^2 \times \mathbb{R})$ on the vectors $d\mathbf{x}/ds \in T_\mathbf{x}(H^2 \times \mathbb{R})$ and $d\mathbf{y}/dt \in T_\mathbf{y}(H^2 \times \mathbb{R})$.

It turns out that the (1,1)-form $\Phi_{1,1}$ obtained in this way is the correct linking integrand on $H^2 \times \mathbb{R}$.

The result

Theorem (with Matt Klein and Lianxin He)

The linking form on $H^2 \times \mathbb{R}$ is

$$\Phi_{1,1} = \frac{1}{8\pi}\left(\frac{(\operatorname{arccosh}\sigma)^2}{((\operatorname{arccosh}\sigma)^2+\tau^2)^{3/2}(\sigma-1)}(dx_3\otimes\omega_{0,1}-\omega_{1,0}\otimes dy_3) + \frac{\tau\operatorname{arccosh}\sigma}{((\operatorname{arccosh}\sigma)^2+\tau^2)^{3/2}\sqrt{\sigma^2-1}}(\alpha_{1,1}+\beta_{1,1})\right)$$

The Heisenberg group

The usual way to think of the Heisenberg group is as the set of 3-by-3 matrices

$$H = \left\{ \begin{bmatrix} 1 & p & r \\ 0 & 1 & q \\ 0 & 0 & 1 \end{bmatrix} \mid p, q, r \in \mathbb{R} \right\},$$

but we will use a slightly different representation via 4-by-4 matrices as follows:

$$H = \left\{ \begin{bmatrix} 1 & a & b & c \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & -a \\ 0 & 0 & 0 & 1 \end{bmatrix} \mid a, b, c \in \mathbb{R} \right\}.$$

Isometries and invariant functions

For the left-invariant metric on the Heisenberg group, the isometry group is the semi-direct product of the group of left translations with the one-dimensional group of rotations in the first two coordinates. If we view H as the subset of $\mathbb{R}^4$ given by $(x_1, x_2, x_3, 1)$, then all isometries can be viewed as linear transformations of $\mathbb{R}^4$ which preserve the hyperplane $x_4 = 1$.

If F is any function of two points $\mathbf{x} = (x_1, x_2, x_3, 1)$ and $\mathbf{y} = (y_1, y_2, y_3, 1)$ invariant under the action of the isometry group, then the derivative of F is zero along Killing fields. Obtain that any isometry-invariant two-point function on H is dependent upon the "common sense" two-dimensional distance function

$$\sigma = (x_1 - y_1)^2 + (x_2 - y_2)^2 \quad \text{and} \quad \tau = y_3 - x_3 - x_1 y_2 + x_2 y_1.$$

Basic differential forms

Since the isometries are unimodular transformations of $\mathbb{R}^4$, we can use the same basic differential forms $\omega_{k,\ell}$ and $\alpha_{k,\ell}$ as before.
Two other families of invariant forms we will need are

$$a_{2,0} = dx_1 \wedge dx_2, \quad a_{1,1} = dx_1 \otimes dy_2 - dx_2 \otimes dy_1, \quad a_{0,2} = dy_1 \wedge dy_2$$

and

$$w_{1,0} = (y_2 - x_2)dx_1 - (y_1 - x_1)dx_2, \qquad w_{0,1} = (y_2 - x_2)dy_1 - (y_1 - x_1)dy_2.$$

Finally, let

$$\begin{aligned} T_{1,1} &= w_{1,0} \otimes w_{0,1} \\ &= (y_2 - x_2)^2 dx_1 \otimes dy_1 - (y_1 - x_1)(y_2 - x_2)(dx_1 \otimes dy_2 + dx_2 \otimes dy_1) \\ &\qquad + (y_1 - x_1)^2 dx_2 \otimes dy_2 \end{aligned}$$

The Key Lemma

Let[1]

$$\Phi_{1,1} = F(\sigma,\tau)(\omega_{1,1} + T_{1,1})$$

and

$$\Psi_{2,0} = F(\sigma,\tau)\omega_{2,0} \qquad \Psi_{0,2} = -F(\sigma,\tau)\omega_{0,2}.$$

We want to choose F so that

$$d_{\mathbf{x}}\Phi_{1,1} = d_{\mathbf{y}}\Psi_{2,0} \quad \text{and} \quad d_{\mathbf{y}}\Phi_{1,1} = d_{\mathbf{x}}\Psi_{0,2}.$$

A calculation yields that we need

$$2\sigma F_\sigma + \tau F_\tau + 3F = 0.$$

And this is a PDE that we can solve!

[1] The form of $\Phi_{1,1}$ given here comes *post facto*, we began by considering a more general form for $\Phi_{1,1}$, namely $F\omega_{1,1} + GT_{1,1} + Ha_{1,1}$, and we learned that we could choose $F = G$ and $H = 0$.

Solutions

We need a solution that is valid for $\sigma \geq 0$ and all τ, and which becomes singular precisely when $\sigma = \tau = 0$, and behaves like $1/\mathrm{distance}^3$ at the singularity. The general solution is

$$F(\sigma, \tau) = \frac{1}{\sigma^{3/2}} \varphi\left(\frac{\tau}{\sqrt{\sigma}}\right)$$

for an arbitrary function φ of one variable.

Solutions which have the proper singularity are:

$$F(\sigma, \tau) = \frac{C}{(A\sigma + B\tau^2)^{3/2}},$$

which comes from choosing $\varphi(x) = C/(A + Bx^2)^{3/2}$.

Since the distance function behaves like $\sqrt{\sigma + \tau^2}$ near $\sigma = \tau = 0$, we choose $A = B = 1$.

Almost there

So our candidate for the linking form on the Heisenberg group is

$$\Phi_{1,1} = \frac{C}{(\sigma + \tau^2)^{3/2}}(\omega_{1,1} + T_{1,1}).$$

To determine C, we need only calculate a single example since we know that the linking formula we obtain is link-homotopy invariant. Let K be the circle given by $\mathbf{x}(s) = (\cos s, \sin s, 0)$ for $s \in [0, 2\pi)$, and let L be the square given in pieces by

$$\begin{aligned}
\mathbf{y}(t) &= (0,0,t) && \text{for} \quad -M \le t < M \\
&= (t-M, 0, M) && \text{for} \quad M \le t < 3M \\
&= (2M, 0, 4M-t) && \text{for} \quad 3M \le t < 5M \\
&= (7M-t, 0, -M) && \text{for} \quad 5M \le t < 7M
\end{aligned}$$

It is easy to show that the contribution to the linking integral from the latter three pieces approach zero as $M \to \infty$, so the linking number of K and L (which is 1) is given by the limit of the linking integral over $K\times$ the first segment as $M \to \infty$.

Theorem (with Matt Klein and Paul Gallagher)

If K and L are disjoint closed curves in H, then

$$\mathrm{Link}(K, L) = \iint_{S^1 \times S^1} \mathbf{X}^* \Phi_{1,1},$$

where

$$\Phi_{1,1} = -\frac{1}{4\pi} \frac{\omega_{1,1} + T_{1,1}}{(\sigma + \tau^2)^{3/2}},$$

The differential form $\Phi_{1,1}$ is invariant under isometries of H.