A Reprise on the Closed Geodesics Problem

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This is a problem for algebraic topology from the beginning: Consider the fundamental groups of oriented surfaces. In the case that the genus is greater than zero, the fundamental group is infinite, and in each homotopy class, there is a closed geodesic.

Let $\mathbb{F}$ denote a field.

The Gromoll-Meyer Theorem (1967). *If M is a compact, closed, manifold of dimension ≥ 2, and the set*

$$\{\dim_{\mathbb{F}} H_i(\Lambda M; \mathbb{F}) \mid i = 0, 1, 2, \ldots\}$$

is unbounded, then infinitely many closed geodesics lie on M in any Riemannian metric.

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Hence there is a Leray-Serre spectral sequence with $E_2^{p,q} \cong H^p(M;\mathbb{F}) \otimes H^q(\Omega M;\mathbb{F})$ and converging to $H^{p+q}(\Lambda M;\mathbb{F})$.

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However, the path-loop fibration has the same E^2-page and converges to $\mathbb{F}$. Hence, the target $H^*(\Lambda M;\mathbb{F})$ lies somewhere between $H^*(M;\mathbb{F}) \otimes H^*(\Omega M;\mathbb{F})$ and $\mathbb{F}$.

Theorem of Sullivan and Vigué-Poirrier (1974). *If X is a finite CW-complex and the cohomology algebra* $H^*(X;\mathbb{Q})$ *requires at least two generators as an algebra, then the set* $\{\dim_{\mathbb{Q}} H_i(\Lambda M;\mathbb{Q}) \mid i = 0, 1, 2, \ldots\}$ *is unbounded.*

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It is a theorem of Borel that $H_*(\Omega V_2(\mathbb{R}^{2k+1});\mathbb{F}_2) \cong \mathbb{F}_2[a_{2k-1}, b_{2k-2}]$ and so it is the case that

$$\{\dim_{\mathbb{F}_2} H_i(\Omega V_2(\mathbb{R}^{2k+1});\mathbb{F}_2) \mid i = 0, 1, 2, \ldots\}$$

is unbounded.

Does this condition hold more generally, that is, suppose $H^*(M; \mathbb{F}_p)$ requires at least two generators as an algebra. Does it follow that

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The answer is **YES**. And the result has several consequences.

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Over a field, the E_2-page of the Eilenberg-Moore spectral sequence is given by

$$E_2^{p,q} \cong \mathrm{Tor}^{p,q}_{H^*(M)\otimes H^*(M)}(H^*(M), H^*(M)).$$

The action of $H^*(M) \otimes H^*(M)$ on $H^*(M)$ is given by a flip and the cup product. To the educated ring theorist, one recognizes

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In the 1960's, Murray Gerstenhaber proved that $HH^*(A)$ enjoys extra structure: It is a graded commutative algebra and, $HH^{*+1}(A)$ is a graded Lie algebra, satisfying

$$[a, bc] = [a, b]c + (-1)^{(|a|-1)|b|} b[a, c].$$

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In our case $H_*(\Lambda M; k) \cong HH^*(C^*(M; k), C^*(M; k))$. Hence, somehow there ought to be a product and Lie bracket on $H_*(\Lambda M; k)$.

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The idea is due to M. Chas and D. Sullivan: Suppose $\alpha\colon \Delta^p \to \Lambda M$ and $\beta\colon \Delta^q \to \Lambda M$ are singular simplices in ΛM. Take the composite

$$\Delta^p \times \Delta^q \to \Lambda M \times \Lambda M \xrightarrow{ev_1 \times ev_1} M \times M$$

and suppose that it is transverse to the diagonal.

At each point where $ev_1 \circ \alpha$ meets $ev_1 \circ \beta$ you have two loops at $\alpha(1) = \beta(1)$. Form the loop product there. This gives a chain

$$\alpha \circ \beta \in C_{p+q-d}(\Lambda M).$$

Theorem. *The chain map* $C_p(\Lambda M) \otimes C_q(\Lambda M) \xrightarrow{\circ} C_{p+q-d}(\Lambda M)$ *induces an associative, commutative algebra structure on* $\mathbb{H}_*(\Lambda M)$.

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1) $(\Lambda M)^{-TM}$ is a homotopy commutative ring spectrum with unit.

2) The product on $(\Lambda M)^{-TM}$ realizes $\circ$ after applying the Thom isomorphism

$$H_q((\Lambda M)^{-TM}) \cong H_{q+d}(\Lambda M) = \mathbb{H}_q(\Lambda M).$$

Theorem (Cohen-Jones-Yan). *If M is an oriented, simply-connected manifold, then there is a 2nd quadrant spectral sequence of algebras* $\{E^r_{p,q}, d^r; p \leq 0, q \geq 0\}$ *such that*

1) $E^r_{*,*}$ *is a bigraded algebra with* $d^r : E^r_{*,*} \to E^r_{*-r,*+r-1}$, *a derivation for each* $r \geq 1$.

2) *The spectral sequence converges to* $\mathbb{H}_*(\Lambda M)$ *as algebras.*

3) *For* $m, n \geq 0$, $E^2_{-m,n} \cong H^m(M; H_n(\Omega M))$ *as algebras, with the product on* $H^*(M)$ *given by the cup product, and the product on* $H_*(\Omega M)$ *given by the Pontryagin product.*

4) *The spectral sequence is natural with respect to smooth maps.*

A classical argument: Consider the CJY spectral sequence for $\mathbb{H}_*(\Lambda V_2(\mathbb{R}^{2k+1}); \mathbb{F}_2)$.

Since $H_*(\Omega V_2(\mathbb{R}^{2k+1}); \mathbb{F}_2) \cong \mathbb{F}_2[a_{2k-1}, b_{2k-2}]$, we can consider the differentials on each sub-polynomial algebra $\mathbb{F}_2[a_{2k-1}]$ and $\mathbb{F}_2[b_{2k-2}]$.

Notice that $d^r(x^2) = 0$ because the algebra is commutative and d^r is a derivation.

Thus, we can apply successive differentials that are zero on successive squares, and hence leave a polynomial algebra on a pair of generators of the form a^{2^j} and b^{2^k}. But a polynomial algebra on two generators has unbounded dimensions. Thus, so does $\mathbb{H}_*(\Lambda M; \mathbb{F}_2)$.

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This argument generalizes depending on the structure of $H_*(\Omega M; \mathbb{F}_p)$.

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A manifold M is *elliptic* mod p if there is an integer $N = N(p)$ and a constant $C = C(p)$ such that

$$\dim_{\mathbb{F}_p} H_r(\Omega M; \mathbb{F}_p) \leq C r^N, \quad r = 1, 2, \ldots$$

A manifold M is *hyperbolic* mod p if there is a constant $K > 1$ such that

$$\sum_{i=0}^{n} \dim_{\mathbb{F}_p} H_i(\Omega M; \mathbb{F}_p) \geq K^{\sqrt{n}}, \text{ for } n \text{ large enough.}$$

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Notice that a compact, oriented manifold M has finite LS-category and the homotopy type of a finite complex, which are assumptions for an elliptic space.

The elliptic case

Theorem (FHT 1991). *If M is an elliptic manifold, then $H_*(\Omega M; \mathbb{F}_p)$ is an elliptic Hopf algebra, and so it is a finitely generated module over a central sub-Hopf algebra which is a polynomial algebra in finitely many indeterminates.*

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In fact, $\Gamma = H_*(\Omega M; \mathbb{F}_p)$ being elliptic may be written as a K-module as $K \otimes G//K$ where K is polynomial and $G//K$ is finite dimensional. The growth of dimensions when $H^*(M; \mathbb{F}_p)$ requires at least two algebra generators implies that K is a polynomial algebra on at least two generators, and hence, the classical argument produces the growth in $\mathbb{H}_*(\Lambda M)$ desired to deduce infinitely many closed geodesics.