Herman Gluck Millersville, October 4, 2013

LIPSCHITZ MINIMALITY
of
GROUP MULTIPLICATION on the THREE-SPHERE

Haomin Wen

The most beautiful maps

between beautiful spaces

ought to be optimal

in some specific mathematical sense,

and then characterized by that optimality.

Haomin's Theorem. *The group multiplication map* $m: S^3 \times S^3 \rightarrow S^3$ *is a Lipschitz constant minimizer in its homotopy class, uniquely so up to composition with isometries of domain and range.*

Remark. The above theorem is easy (and fun) to prove for S^1 .

Haomin's proof for S^3 also works for the multiplication map $m: S^7 \times S^7 \rightarrow S^7$ of unit Cayley numbers.

Lipschitz maps and constants.

A map $f: X \to Y$ between metric spaces is a ***Lipschitz map*** if there is a constant C such that $d(f(x), f(x')) \leq C\, d(x, x')$ for all x, x' in X.

The smallest such constant C is called the ***Lipschitz constant*** of f.

There always exists a Lipschitz constant minimizer in the homotopy class of any Lipschitz map between compact metric spaces (by Arzela-Ascoli).

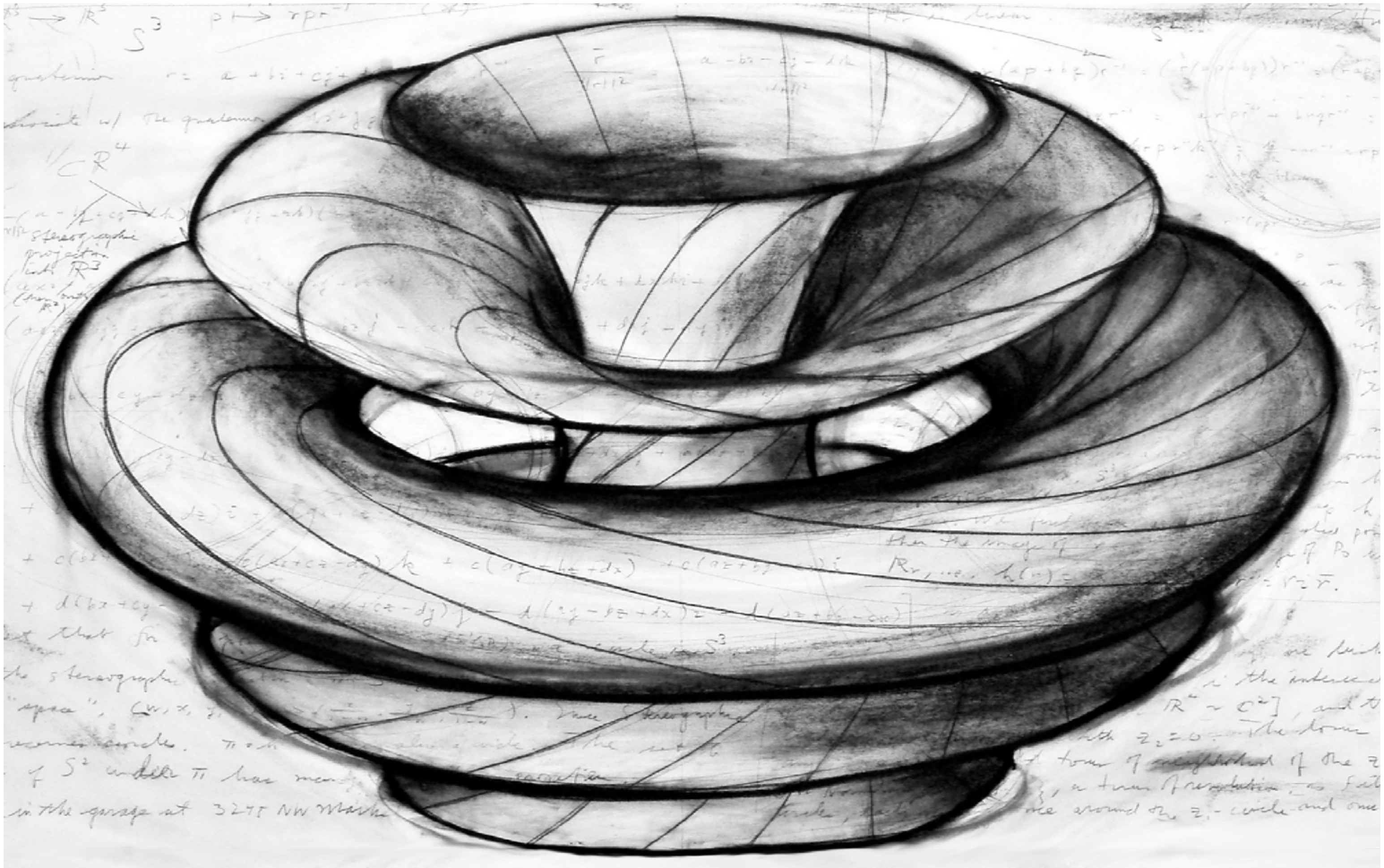
Background to Haomin's theorem. Consider the Hopf fibrations of round spheres by parallel great subspheres:

$S^1 \subset S^3 \to S^2 = CP^1$, $S^1 \subset S^5 \to CP^2$, ..., $S^1 \subset S^{2n+1} \to CP^n$, ...
$S^3 \subset S^7 \to S^4 = HP^1$, $S^3 \subset S^{11} \to HP^2$, ..., $S^3 \subset S^{4n+3} \to HP^n$, ...
$S^7 \subset S^{15} \to S^8$,

with the nonassociativity of the Cayley numbers responsible for the truncation of the third series.

First one discovered by Hopf in 1931, rest by him in 1935.

All Hopf projections have Lipschitz constant 1 when the base spaces are given the Riemannian submersion metric.



Hopf fibration of 3-sphere by great circles
Lun-Yi Tsai   Charcoal and graphite on paper   2007

Thms (with Dennis DeTurck and Pete Storm, 2010).

(1) *Given a Hopf fibration of a round sphere by parallel great subspheres, the projection map to the base space is, up to isometries of domain and range, the unique Lipschitz constant minimizer in its homotopy class.*

Dennis DeTurck

(2) *When the fibres of a Hopf fibration are great circles, a unit vector field tangent to these circles is, up to isometries of domain and range, the unique Lipschitz constant minimizer in its homotopy class.*

Pete Storm

Tracing even further back...

Theorem (with Wolfgang Ziller, 1986).
On S^3, the Hopf vector field is volume-minimizing in its homology class in the unit tangent bundle.

Wolfgang Ziller

However ... **(David Johnson, 1988).**

On S^5 *, the Hopf vector field is* ***not*** *volume-minimizing in its homology class, not even a local minimum, though it is a critical "point" of the volume function.*

We are **HOPING** that many beautiful maps...for example, Riem. submersions of compact homogeneous spaces... can be shown to be Lipschitz minimizers in their homotopy classes, unique up to composition with isometries of domain and range.

The Hopf projections all have this feature.

One more known instance. The Stiefel projection $V_2R^4 \rightarrow G_2R^4$ is a Lipschitz constant minimizer in its homotopy class, unique up to composition with isometries of domain and range.

Remark. Group multiplication $S^3 \times S^3 \rightarrow S^3$ is, up to scale, a Riemannian submersion of compact homogeneous spaces.

In the following pages, we will display the architecture of Haomin's proof of his theorem, but give not details.

Back to S^3 with a preliminary result.

(1) Group multiplication $S^3 \times S^3 \rightarrow S^3$ has Lipschitz constant $= \sqrt{2}$.

This is a matter of observation, which we tackle in a moment.

(2) Any map $S^3 \times S^3 \rightarrow S^3$ homotopic to this has Lipschitz constant $\geq \sqrt{2}$.

(1) Group mult $m: S^3 \times S^3 \to S^3$ has $Lip(m) = \sqrt{2}$.

Proof. In a Lie group with bi-invariant metric, group mult near all pairs of points are isometric.

Enough to show the differential $m_* : R^3 \times R^3 \to R^3$ has Lipschitz constant (= operator norm) $\sqrt{2}$.

At (identity, identity), m_* = addition in R^3 .

The matrix A of addition is the 3×6 matrix $\mathbf{I} \mid \mathbf{I}$.

$Lip(A) = ||A||_{op} = \sqrt{(\text{largest eigenvalue of } A^T A)}$

The eigenvalues of $A^T A$ are computed to be 0, 0, 0, 2, 2, 2 , completing the proof.

Preliminaries to the proof of (2).

Definitions. A map $f : S^n \to S^n$ is said to be

even if $f(-x) = f(x)$ for all $x \in S^n$;

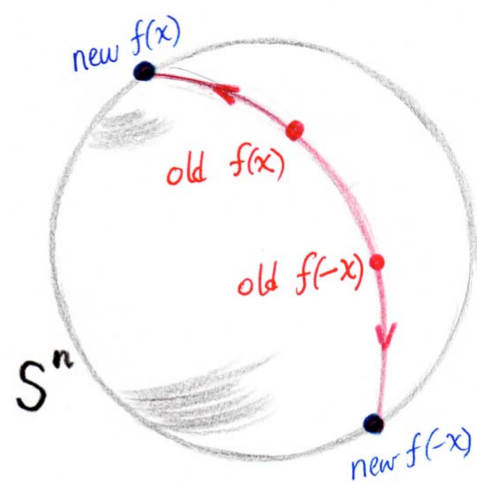
odd if $f(-x) = -f(x)$ for all $x \in S^n$.

Easy exercise. An even map $S^n \to S^n$ has even degree.

Theorem (Borsuk). An odd map $S^n \to S^n$ has odd degree. (For a proof, see Hatcher, "Algebraic Topology," pp. 174-176.

Corollary 1. If $f : S^n \to S^n$ has even degree, then there is a pair of antipodal points x and $-x$ such that $f(x) = f(-x)$.

Proof. Suppose not. Then homotope f by repulsion so that afterwards $f(-x) = -f(x)$ for *every* x in S^n .

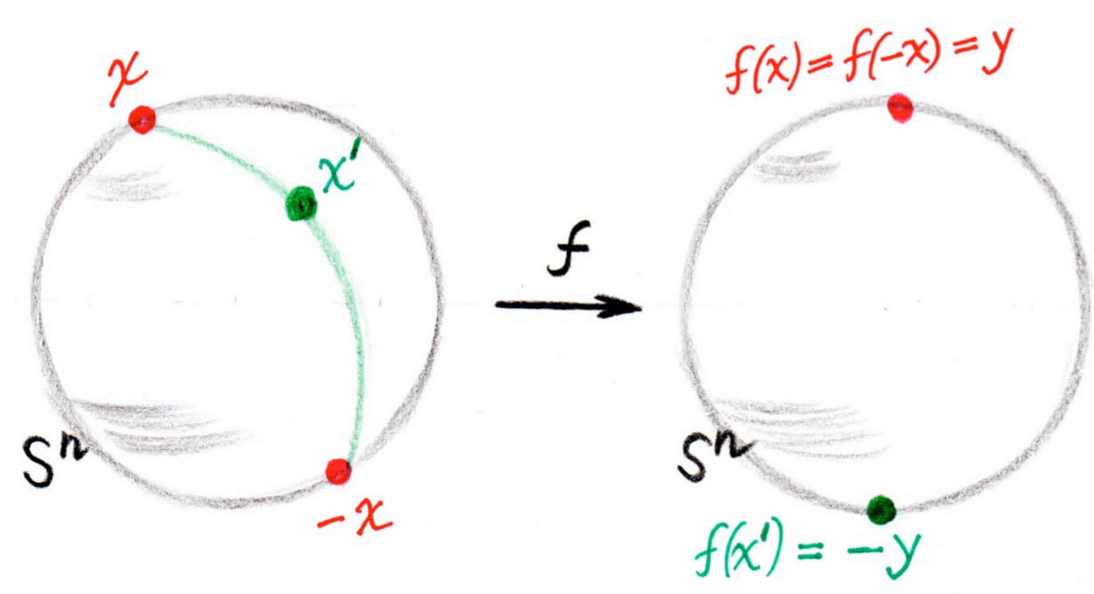


By Borsuk's Theorem, this implies that f has odd degree, contrary to assumption.

Corollary 2. A degree-two map $f : S^n \rightarrow S^n$ must have Lipschitz constant ≥ 2 .

Proof. By Corollary 1, there exists a pair of antipodal points x and $-x$ such that $f(x) = f(-x) =$ some y .

Let x' be a point in S^n such that $f(x') = -y$.



Then $d(x', x) \leq \pi/2$ or $d(x', -x) \leq \pi/2$,
yet $d(f(x'), f(x)) = d(f(x'), f(-x)) = \pi$.

Hence $\mathrm{Lip}(f) \geq 2$.

Proof of (2): Any map $f: S^3 \times S^3 \to S^3$ which is homotopic to the multiplication map $m: S^3 \times S^3 \to S^3$ has $\mathrm{Lip}(f) \geq \sqrt{2}$.

The restriction of m to the diagonal

$$\Delta(S^3) = \{(x, x): x \in S^3\} \to S^3$$

has degree 2 , so the same must hold for f .

Since $\Delta(S^3)$ is a round 3-sphere of radius $\sqrt{2}$, it follows from Corollary 2 that $\mathrm{Lip}(f|_{\Delta(S^3)}) \geq \sqrt{2}$.

Hence $\mathrm{Lip}(f) \geq \sqrt{2}$, as claimed.

Remark. At this point, we know that the multiplication map

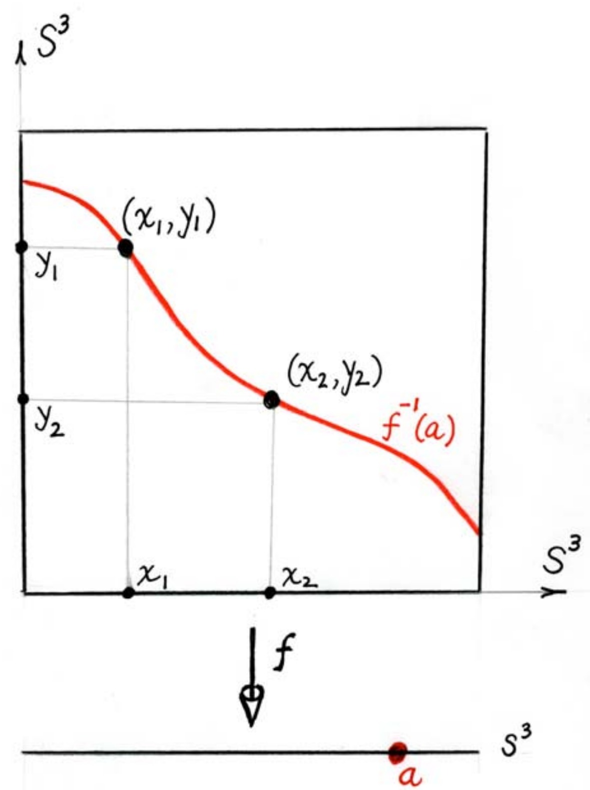
$$m: S^3 \times S^3 \to S^3$$

has the minimum possible Lipschitz constant of $\sqrt{2}$ in its homotopy class.

The issue now is to show that the only other maps in this homotopy class with Lipschitz constant $\sqrt{2}$ are the compositions of m with isometries of domain and range.

The four steps of Haomin's proof of uniqueness.

To start, let (x_1, y_1) and (x_2, y_2) be two pts in $S^3 \times S^3$ which have the same image in S^3 under f.



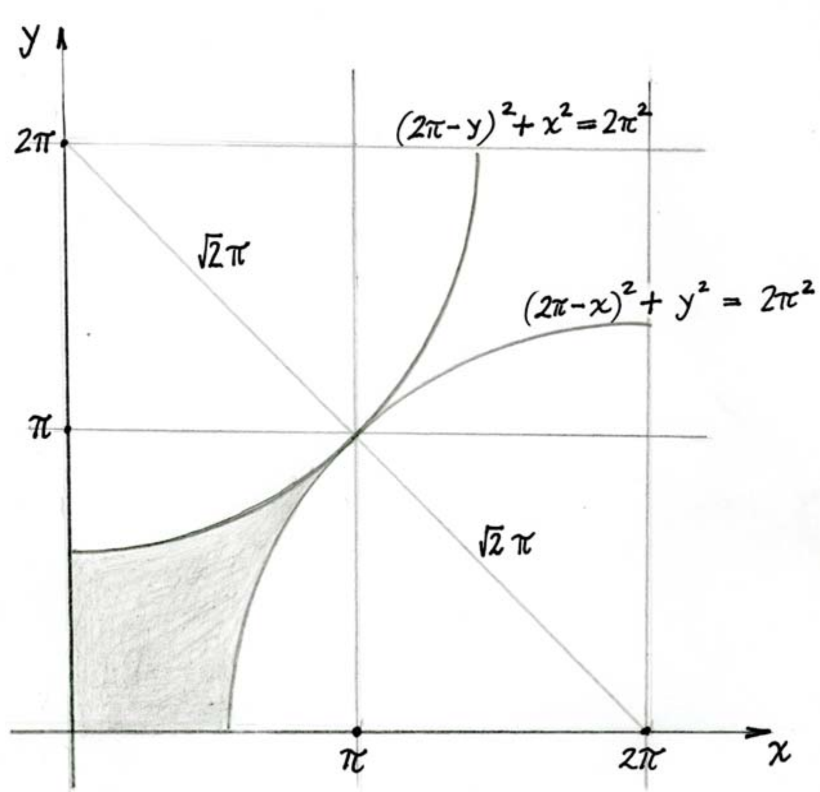
Step 1. Haomin proves the following inequalities:

$$(2\pi - d(x_1, x_2))^2 + d(y_1, y_2)^2 \geq 2\pi^2$$

$$(2\pi - d(y_1, y_2))^2 + d(x_1, x_2)^2 \geq 2\pi^2 ,$$

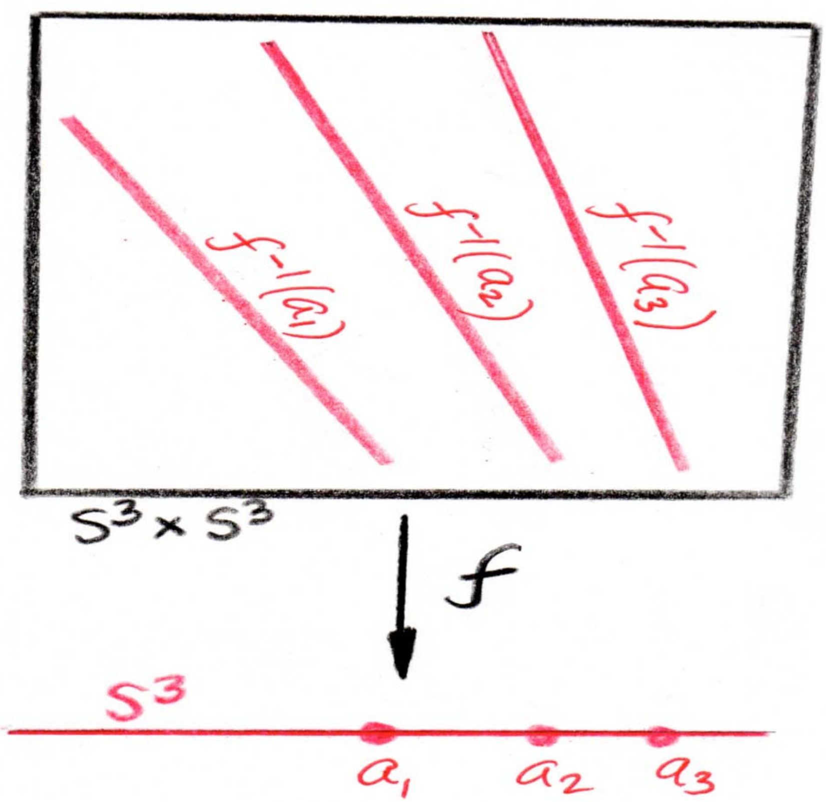
which are at the heart of his argument.

We graph both inequalities together in the figure below, letting $x = d(x_1, x_2)$ and $y = d(y_1, y_2)$, both in $[0, \pi]$.

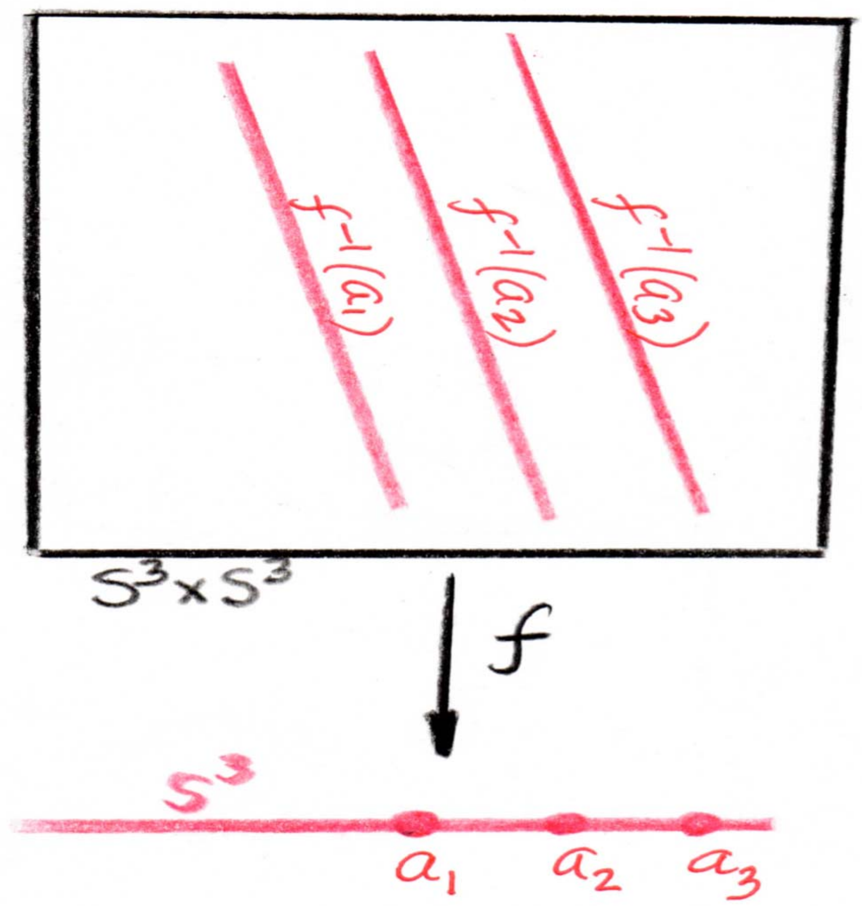


The shaded region above consists of the points (x, y) satisfying both inequalities.

Step 2. Haomin uses these inequalities to show that each inverse image $f^{-1}(a)$ is the graph of some isometry $h_a: S^3 \rightarrow S^3$, and hence appears inside $S^3 \times S^3$ as a diagonal 3-sphere.



Step 3. Haomin shows that these diagonal 3-spheres are parallel to one another.



Step 4. Haomin uses classical results to finish the proof.

Proposition (Y-C Wong 1961, Joseph Wolf 1963). *Any fibration of an open set on* $S^3 \times S^3$ *by parallel great 3-spheres extends to a fibration of all of* $S^7(\sqrt{2})$ *by parallel great 3-spheres, and any two of these are isometric to one another.*

It follows that any two fibrations of $S^3 \times S^3$ by parallel great 3-spheres can be taken, one to the other, by an isometry of $S^3 \times S^3$.

Perform this isometry, so that now the fibres of $f: S^3 \times S^3 \to S^3$ coincide with the fibres of the multiplication map $m: S^3 \times S^3 \to S^3$.

Since f and m are now both Riemannian submersions (up to scale) of $S^3 \times S^3 \to S^3$ having the same fibres, the map of S^3 to itself which takes $f(x, y)$ to $m(x, y)$ is an isometry.

This completes the proof of Haomin's theorem.

What's next?

Test question #1: Try to show that the bundle map

$$SO(n) \to S^{n-1}$$

is a Lipschitz constant minimizer in its homotopy class, unique up to composition with isometries of domain and range.

This can be shown for $n \leq 4$ on the basis of known results, so the first challenge is for $n = 5$.

Test question #2: Show that the projection map

$$SU(3) \rightarrow S^5$$

is a Lipschitz constant minimizer in its homotopy class, unique up to composition with isometries of domain and range.

Test question #3: Show that the projection map of the Stiefel bundle

$$V_2R^n \rightarrow G_2R^n$$

is a Lipschitz constant minimizer in its homotopy class, unique up to composition with isometries of domain and range.

This is also known for $n \leq 4$, so the first challenge is for $n = 5$.

Test question #4: Geometry of real Grassmann mflds.

Let $G_k R^n$ = set of oriented k-planes thru origin in R^n .

$G_k R^n = SO(n) / (SO(k) \times SO(n-k))$ = k(n – k) dim'l mfld.

$$
\begin{array}{ccccccccccc}
G_5R^6 & \subset & G_5R^7 & \subset & G_5R^8 & \subset & G_5R^9 & \subset & G_5R^{10} & \subset & \ldots \\
\cup & & \cup & & \cup & & \cup & & \cup & & \\
G_4R^5 & \subset & G_4R^6 & \subset & G_4R^7 & \subset & G_4R^8 & \subset & G_4R^9 & \subset & \ldots \\
\cup & & \cup & & \cup & & \cup & & \cup & & \\
G_3R^4 & \subset & G_3R^5 & \subset & G_3R^6 & \subset & G_3R^7 & \subset & G_3R^8 & \subset & \ldots \\
\cup & & \cup & & \cup & & \cup & & \cup & & \\
G_2R^3 & \subset & G_2R^4 & \subset & G_2R^5 & \subset & G_2R^6 & \subset & G_2R^7 & \subset & \ldots \\
\cup & & \cup & & \cup & & \cup & & \cup & & \\
G_1R^2 & \subset & G_1R^3 & \subset & G_1R^4 & \subset & G_1R^5 & \subset & G_1R^6 & \subset & \ldots
\end{array}
$$

$$\begin{array}{ccccc} G_3R^4 & \subset & G_3R^5 & \subset & G_3R^6 \\ \cup & & \cup & & \cup \\ G_2R^3 & \subset & G_2R^4 & \subset & G_2R^5 \\ \cup & & \cup & & \cup \\ G_1R^2 & \subset & G_1R^3 & \subset & G_1R^4 \end{array}$$

The 9-dim'l Grassmann manifold G_3R^6 has the rational homotopy type of $S^4 \times S^5$, and the subGrassmannian G_2R^4 generates its 4-dim'l homology.

But (with Dana Mackenzie and Frank Morgan, 1995) ... G_2R^4 is only a local volume-minimizer in its homology class in G_3R^6, not a global volume-minimizer.

Test question #4. Is the inclusion of G_2R^4 in G_3R^6 a Lipschitz minimizer in its homotopy class?