Quantum Bits and Quaternions

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Outline

Classical and quantum computers

bitstring ⟶ classical processor ⟶ bitstring

quantum state ⟶ quantum processor ⟶  measurement ⟶ bitstring

Classical computer, more detail

bitstring	$\longrightarrow$	classical processor	$\longrightarrow$	bitstring
$x \in \{0,1\}^n$	$\longrightarrow$	Boolean function f	$\longrightarrow$	$f(x)$

$$x \xrightarrow{f} y$$

Quantum computer, more detail

quantum state $\longrightarrow$ quantum processor $\longrightarrow$ measurement $\longrightarrow$ bitstring

complex vector $\sum_{x\in\{0,1\}^n} c_x \mathbf{e}_x$ $\longrightarrow$ unitary operator $\longrightarrow$??? $\rightsquigarrow$ $x \in \{0,1\}^n$

$$\sum_x c_x \mathbf{e}_x \xrightarrow{U} \sum_x a_x \mathbf{e}_x \rightsquigarrow$$

x with prob. $|a_x|^2$

Comparison: classical vs quantum computers

classical: $x \xrightarrow{f} y$

quantum: $\sum_x c_x \mathbf{e}_x \xrightarrow{U} \sum_x a_x \mathbf{e}_x$  $\rightsquigarrow$ x with prob. $|a_x|^2$

- exponential speedup for the quantum processor
- cost of quantum state preparation
- cost of quantum circuit preparation
- cost of readout
- cost of error correction

The dream: processor speedup will outweigh the costs
The fine print: many qualifications, much commentary, non-trivial controversy

Efficient Circuit Design

- "Circuit" = composition of unitary operators (LARGE)
- "Gate" = unitary operator (SMALL)
- Need standardized gates to build larger circuits
- Some unitary gates are more expensive than others
- Desired: small set of cheap, versatile gates that combine efficiently to implement many circuits
- "Universal Gate Set" = set of gates that be combined to approximate any needed unitary

Natural Thing to Want

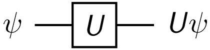
A universal gate set that optimizes a bunch of practical considerations

Example gate: 1-qubit unitary

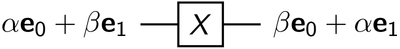
- A state of a quantum bit (qubit) is a vector $\begin{bmatrix}\alpha \\ \beta\end{bmatrix} = \alpha\mathbf{e}_0 + \beta\mathbf{e}_1 \in \mathbb{C}^2$
- A 1-qubit unitary is simply at 2×2 unitary matrix U
- $U = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ acts on the state $\begin{bmatrix}\alpha \\ \beta\end{bmatrix} = \alpha\mathbf{e}_0 + \beta\mathbf{e}_1$ by

$$\begin{bmatrix}\alpha \\ \beta\end{bmatrix} \xrightarrow{U} \begin{bmatrix} a\alpha + b\beta \\ c\alpha + d\beta \end{bmatrix}$$

- Circuit diagram for 1-qubit unitary

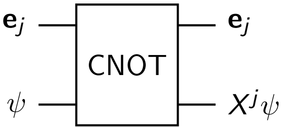


- Example: $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ ("Quantum NOT gate")



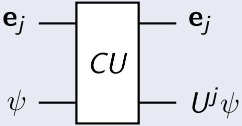
Example gate: CNOT gate

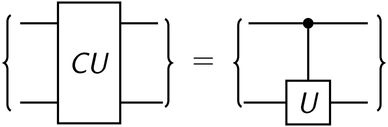
- $\text{CNOT} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$
- $\alpha \mathbf{e}_{00} + \beta \mathbf{e}_{01} + \gamma \mathbf{e}_{10} + \delta \mathbf{e}_{11} \xrightarrow{\text{CNOT}} \alpha \mathbf{e}_{00} + \beta \mathbf{e}_{01} + \delta \mathbf{e}_{10} + \gamma \mathbf{e}_{11}$
- $\mathbf{e}_i \mathbf{e}_j = \mathbf{e}_{ij}$ $(\mathbf{e}_i \mathbf{e}_j = \mathbf{e}_i \otimes \mathbf{e}_j)$
- $\alpha \mathbf{e}_{00} + \beta \mathbf{e}_{01} + \gamma \mathbf{e}_{10} + \delta \mathbf{e}_{11} = \mathbf{e}_0(\alpha \mathbf{e}_0 + \beta \mathbf{e}_1) + \mathbf{e}_1(\gamma \mathbf{e}_0 + \delta \mathbf{e}_1)$
- $\mathbf{e}_j \psi \xrightarrow{\text{CNOT}} \mathbf{e}_j X^j \psi$

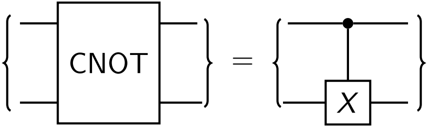


Controlled gates

For a 1-qubit unitary U, the controlled-U gate (CU) is







Controlled gates, cont'd

A universal gate set

The set of 1-qubit unitaries, together with the CNOT gate

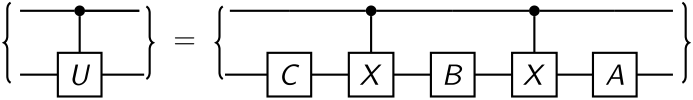
Problem

Implement a controlled-U using only 1-qubit unitaries and CNOT

Unitary factorization that gives a solution

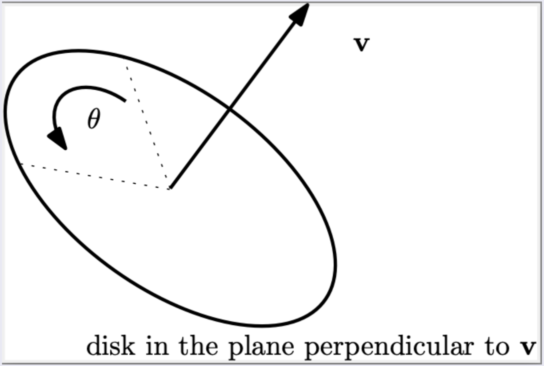
Given a 1-qubit unitary U, there exist 1-qubit unitaries A, B, C such that

$$U = AXBXC$$
$$\mathrm{Id} = ABC$$



Rotations of $\mathbb{R}^3$

The rotation $R_{\mathbf{v},\theta}$ about the vector $\mathbf{v}$ by the angle θ



The rotation group

$$\mathrm{Rot}(\mathbb{R}^3) = \{R_{\mathbf{v},\theta} : \mathbf{v} \in \mathbb{R}^3, \theta \in \mathbb{R}\}$$

A good way to implement rotations

- The quaternions $\mathbb{H} = \mathbb{R}^4$ as real vector space
- Let $\mathcal{M} = \left\{ \begin{bmatrix} u & v \\ -v^* & u^* \end{bmatrix} : u, v \in \mathbb{C} \right\}$
- Easy exercise: $\mathcal{M}$ is a real algebra (closed under matrix mult.)
- Use these identifications to put multiplication on $\mathbb{H}$

$$\mathbb{H} \leftrightarrow \mathbb{C}^2 \leftrightarrow \mathcal{M}$$

$$(a, b, c, d) \leftrightarrow (\underbrace{a + bi}_{u}, \underbrace{c + di}_{v}) \leftrightarrow \begin{bmatrix} u & v \\ -v^* & u^* \end{bmatrix}$$

-

$$\mathbf{1} = (1, 0, 0, 0) \leftrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \mathbf{i} = (0, 1, 0, 0) \leftrightarrow \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

$$\mathbf{j} = (0, 0, 1, 0) \leftrightarrow \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \mathbf{k} = (0, 0, 0, 1) \leftrightarrow \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

A good way to implement rotations, cont'd

Unit quaternions as rotations

- let $r = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ be a unit quaternion $(a^2 + b^2 + c^2 + d^2 = 1)$
- let $u = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ be a pure quaternion
- Let $R_r(u) = rur^*$ $\quad (r^* = a - b\mathbf{i} - c\mathbf{j} - d\mathbf{k})$
- R_r is a rotation of $\mathbb{R}^3$

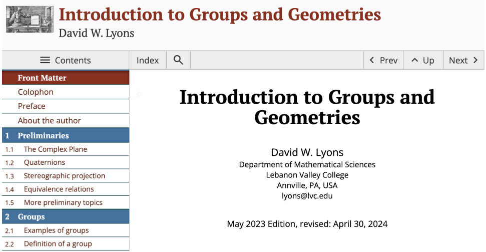
$$R_r(u) = R_{\mathbf{v},\theta}(x, y, z)$$
$$\mathbf{v} \propto (b, c, d)$$
$$\theta = 2\cos^{-1}(a)$$

Unit quaternions and unitaries

The set of unit quaternions identifies with the group $SU(2) \subseteq \mathcal{M}$. This gives a natural way to think about unitary operators as rotations.

Introduction to Groups and Geometries
mathvista.org
lyons@lvc.edu

- Combines introductory group theory with modern geometries
- Serves a wide audience of math and physics students
- Provides useful language and problem solving tools



A Rotation Decomposition

Task for a superhero

- Space monster rotates the earth out of kilter
- Superhero (you) must put the earth back right
- Use only rotations about two axes

Solution

inverse of the space monster's rotation is

$$R = R_{Z,\theta_3} \circ R_{Y,\theta_2} \circ R_{Z,\theta_1}$$

Consequence

Any rotation of 3-space can be implemented by composing at most three rotations about the Z and Y axes

Another rotation decomposition

Find rotations A, B, C so that $R = AXBXC$ and $ABC = \mathrm{Id}$

$$
\begin{aligned}
X &= R_{X,\pi} \\
R &= R_{Z,\gamma} \circ R_{Y,\beta} \circ R_{Z,\alpha} \\
&= R_{Z,\gamma} \circ \left(R_{Y,\frac{\beta}{2}} \circ R_{Y,\frac{\beta}{2}}\right) \circ \left(R_{Z,\frac{\alpha+\gamma}{2}} \circ R_{Z,\frac{\alpha-\gamma}{2}}\right) \\
&= \underbrace{(R_{Z,\gamma} \circ R_{Y,\frac{\beta}{2}})}_{A} \circ R_{Y,\frac{\beta}{2}} \circ R_{Z,\frac{\alpha+\gamma}{2}} \circ \underbrace{R_{Z,\frac{\alpha-\gamma}{2}}}_{C} \\
&= \underbrace{(R_{Z,\gamma} \circ R_{Y,\frac{\beta}{2}})}_{A} \circ \left(X \circ R_{Y,-\frac{\beta}{2}} \circ X\right) \circ \left(X \circ R_{Z,\frac{-(\alpha+\gamma)}{2}} \circ X\right) \circ \underbrace{R_{Z,\frac{\alpha-\gamma}{2}}}_{C} \\
&= \underbrace{(R_{Z,\gamma} \circ R_{Y,\frac{\beta}{2}})}_{A} \circ X \circ \underbrace{(R_{Y,-\frac{\beta}{2}} \circ R_{Z,\frac{-(\alpha+\gamma)}{2}})}_{B} \circ X \circ \underbrace{R_{Z,\frac{\alpha-\gamma}{2}}}_{C} \\
&= AXBXC
\end{aligned}
$$

Why did we just do that?

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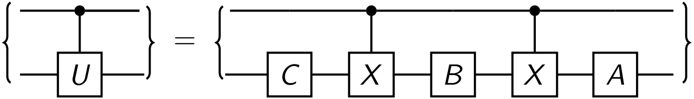
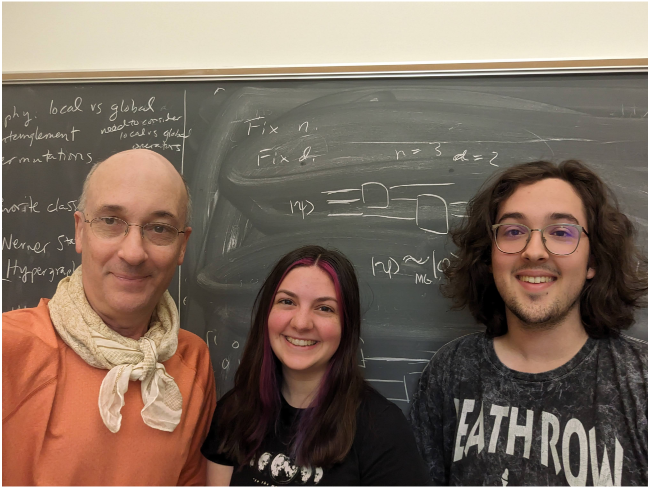


Summary and Outlook

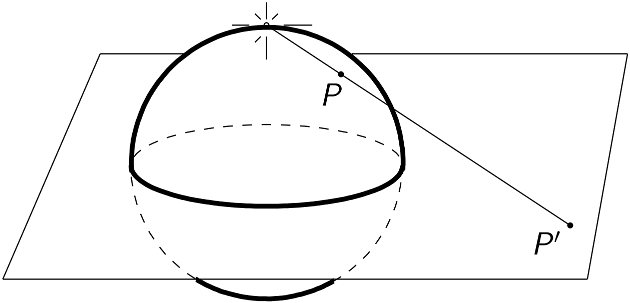
- Quantum computation is promising for practical applications
- Quantum computation is filled with a lot of great math problems
- Many of those problems have few prerequisites, however . . .
- At least some facility with basics of abstract algebra and modern geometry is necessary
- For the sake of time and productivity, introductions to algebra and geometry must be streamlined and efficient
- The quaternions play a useful role, hurray!
- Thoughtful attention to background training makes real research accessible to undergraduates

Thank you!



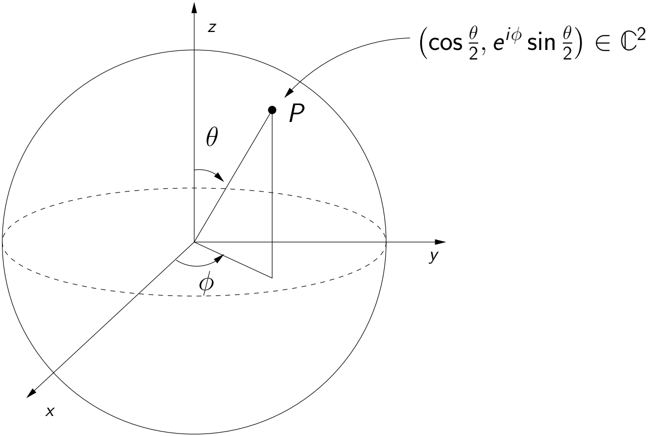
http://quantum.lvc.edu/mathphys

Stereographic projection $S^2 \to \mathbb{C} \cup \{\infty\}$



$$P = (a, b, c) \to \frac{a + ib}{1 - c} = P'$$

Bloch sphere



For $(a, b) \in \mathbb{C}^2$, we have $\text{Bloch}(a, b) = \mathrm{stereo}^{-1}\overline{(a/b)}$

Comparison of rotation conventions

Ways that $(a, b, c, d) \in S^3$ acts as a rotation on S^2

$$
\begin{array}{lll}
\text{(unit quaternions)} & \leftrightarrow \text{(Möbius elliptic group)} & \leftrightarrow SU(2) \\
a + bi + cj + dk & \leftrightarrow \left[z \to \frac{(a+bi)z+(c+di)}{(-c+di)z+(a+bi)} \right] & \leftrightarrow \begin{bmatrix} a+bi & c+di \\ -c+di & a-bi \end{bmatrix}
\end{array}
$$

	axis	angle
quaternions	(b, c, d)	$2\cos^{-1} a$
Möbius transf.	$(d, -c, b)$	$2\cos^{-1} a$
Bloch coordinates	(d, c, b)	$-2\cos^{-1} a$

References