Pythagorean Triples

MAS 100 Concepts of Mathematics
Sample Paper
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Everyone is familiar with the Pythagorean Theorem, which states that if a and b are the lengths of the legs of a right triangle and c is the length of the hypotenuse, then $a^2+b^2=c^2$. Also familiar is the "3-4-5" right triangle. It is an example of three *whole* numbers that satisfy the equation in the theorem. Other such examples, from less familiar to downright obscure, are right triangles with side lengths 5-12-13, 35-12-37, and 45-28-53. Why are these interesting? Well, if you try to look for right triangles with whole number side lengths by trial and error, you will mostly be frustrated, since most of the time two whole number side lengths go with an irrational third side, like 1-1-$\sqrt{2}$. Is there a pattern to the whole number triples? If so, what is it?

> **Problem A.** Describe all possible triples (called *Pythagorean triples*) of whole numbers (a, b, c) which can occur as the lengths of the three sides of a right triangle. In other words, find all positive whole number solutions to the equation $a^2 + b^2 = c^2$.

A complete solution was known to the Greeks by about 500 BCE ([3], p.37). Weisstein gives a thorough survey of solutions to this and related problems on his *MathWorld* website [4]. In this paper we describe a solution which uses a beautiful map called *stereographic projection.* Similar presentations can be found in [1] and [2].

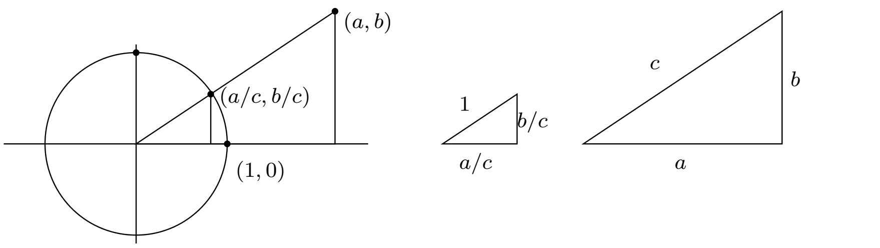

Figure 1: Pythagorean triple (a, b, c) and associated unit circle point $(a/c, b/c)$

First we make an observation that transforms the problem to a question about points on the unit circle. If (a, b, c) is a Pythagorean triple, we can divide both sides of $a^2 + b^2 = c^2$ by c^2 to obtain $(a/c)^2 + (b/c)^2 = 1$. So a Pythagorean triple (a, b, c) gives rise to a point $(a/c, b/c)$ on the unit circle in the first quadrant, both of whose coordinates are rational. See Figure 1. Conversely, suppose we have a point P on the unit circle in the first quadrant with rational x and y coordinates, say $P = (m/n, p/q)$. Multiplying both sides of $(m/n)^2+(p/q)^2 = 1$ by n^2q^2, we obtain $(mq)^2+(pn)^2 = (nq)^2$, so (mq, pn, nq) is a Pythagorean triple. This correspondence between Pythagorean triples and points on the unit circle in the first quadrant with rational coordinates is *not* one-to-one. For example, the triples $(3, 4, 5)$ and $(6, 8, 10)$ both correspond to the unit circle point $(3/5, 4/5)$. Indeed, for any triple (a, b, c), all of its multiples (ka, kb, kc) share the same unit circle point $(a/c, b/c)$. A Pythagorean triple (a, b, c) is called *reduced* if a, b, c share no common factors. The advantage of considering reduced triples is that each circle point with rational coordinates corresponds to exactly one reduced triple. Thus we arrive at a new version of the problem.

> **Problem B.** Describe all points on the unit circle in the first quadrant with rational coordinates.

If we can solve problem B, then problem A is solved by these steps.

1. For each unit circle point $(m/n, p/q)$, generate a triple (mq, pn, nq).
2. Remove common factors (if there are any) to produce a reduced triple (a, b, c).
3. Now we have an infinite family of triples which are positive whole number multiples (ka, kb, kc) of the reduced triple (a, b, c). In this way we account for all possible triples.

But how do we solve problem B? The key is stereographic projection which we describe in the next paragraph.

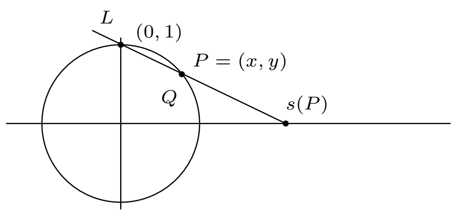

Figure 2: Stereographic projection

Let Q be the first quadrant of the unit circle in the x, y-plane. Given a point $P = (x, y)$ on Q, let L be the line through P and $(0, 1)$. Let $s(P)$ be the x-coordinate of the intersection of L with the x axis. See Figure 2. This defines a one-to-one correspondence $s\colon Q \to (1, \infty)$ called stereographic projection. By inspection of the similar triangles in Figure 3, we see that the formula for s is

$$s(x, y) = \frac{x}{1 - y}.$$

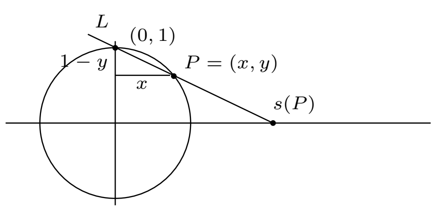

Figure 3: Similar triangles yield a formula for $s(P)$

To find a formula for s^{-1}, we solve the pair of equations $x/(1 - y) = a$ and $x^2 + y^2 = 1$ for x and y in terms of a. We obtain

$$s^{-1}(a) = \left(\frac{2a}{a^2 + 1}, \frac{a^2 - 1}{a^2 + 1} \right).$$

From these expressions, it is apparent that s takes points with rational coordinates to rational numbers, and s^{-1} takes rational numbers to points with rational coordinates. Thus we arrive at yet another version of the problem.

Problem C. Describe the set of rational numbers greater than 1.

A straightforward solution to problem C is the following list in "dictionary order" of all reduced fractions $m/n > 1$.

$$2/1, 3/1, 3/2, 4/1, 4/3, 5/1, 5/2, 5/3, 5/4, \ldots.$$

Dictionary order means that you read the expression m/n from left to right like a "word." The word m/n comes before the word p/q if $m < p$ or if $m = p$ and $n < q$ (just the same rules as for words in a dictionary). *Reduced* means that m, n share no common factors. Notice $m > n$ for each of the fractions since $m/n > 1$.

Here's how to put all these observations together to solve the problem of Pythagorean triples. Beginning with the first rational 2/1 in the above list, use inverse stereographic projection to find the unit circle point $(4/5, 3/5)$. This corresponds to the reduced Pythagorean triple $(4, 3, 5)$, which in turn has an infinite family $(4k, 3k, 5k)$ of all of its positive whole number multiples. Repeating this process as we move down the list reduced rationals greater than 1 accounts for all possible Pythagorean triples. The following table shows the first few steps of the process.

rational $a = m/n$	circle point $s^{-1}(a)$	reduced triple (a, b, c)
$2/1$	$(4/5, 3/5)$	$(4, 3, 5)$
$3/1$	$(6/10, 8/10)$	$(3, 4, 5)$
$3/2$	$(12/13, 5/13)$	$(12, 5, 13)$
$4/1$	$(8/17, 15/17)$	$(8, 15, 17)$
$4/3$	$(24/25, 7/25)$	$(24, 7, 25)$
$\vdots$	$\vdots$	$\vdots$

Table 1: Ordered list of reduced Pythagorean triples

We have shown how finding integer solutions to the equation $a^2 + b^2 = c^2$ can be rephrased as a problem in analytic geometry, namely, finding points on the unit circle with rational coordinates. We demonstrated how this problem is solved by means of explicit formulas for stereographic projection and its inverse, and how the description of rational points on the circle translates into a description of Pythagorean triples.

References

[1] Alexander Bogomolny, Pythagorean Triples, http://www.cut-the-knot.org/pythagoras/pythTriple.shtml

This website has a large collection of mathematical recreation (puzzles and games) and also educational topics.

[2] William Casselman, The parametrization of Pythagorean triples, http://www.math.ubc.ca/~cass/courses/m446-03/pl322/parametrization.html

Casselman is a mathematics professor at the University of British Columbia. One of his special interests is visualizing mathematical objects.

[3] Howard Eves, *An Introduction to the History of Mathematics, 4th edition*, Holt, Rinehart and Winston, New York, 1976

> This book gives a history of mathematics from the first written records to the 20th century. It tells the story of ancient Greek mathematics and the Pythagorean school in particular. Eves has written many books on mathematics history.

[4] Eric Weisstein, Pythagorean Triple, http://mathworld.wolfram.com/PythagoreanTriple.html

> This website is maintained by the creator of the math software Mathematica. It gives basic explanations and examples on many topics.